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Parallel Homotopy Continuation in Julia

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Contents

1	Introduction	2
2	Homotopy Continuation	2
2.1	Choosing the homotopy	2
2.1.1	Gamma trick	3
2.2	Tracking down the roots	3
2.2.1	Predictor: Euler's method	4
2.2.2	Corrector: Newton's method	4
2.2.3	Adaptive step size	4
3	Testing the method	5
4	Appendix A: Results	5
4.1	Single- vs Multi-threaded	5
4.2	Parallelization	7
5	Appendix B: Implementation	7
5.1	Julia code	7
5.2	Hardware	11

1 Introduction

Homotopy Continuation is a numerical method for solving systems of polynomial equations. It is based on the idea of "deforming" a given system of equations into a simpler one whose solutions are known, and then tracking the solutions of the original system as the deformation is undone.

In this project, the method will be implemented in the Julia programming language, which is particularly suited for scientific computing. The primary source for this report is [1], where the method is explained in much more detail.

2 Homotopy Continuation

We will only consider *square* systems of polynomial equations, i.e. systems of n polynomial equations in n variables, although over- or under-determined systems can often be solved by reducing them to square systems, by respectively choosing a suitable square subsystem or squaring it by adding equations. Moreover, we will restrict ourselves to systems which have isolated solutions, i.e. zero-dimensional varieties.

There are many ways to choose the "simpler" system, from now on called a *start system*, but in general we can observe that, by Bezout's theorem, a system $F = (f_1, \dots, f_n)$ has at most $D := d_1 \dots d_n$ solutions, where d_i is the degree of $f_i(x_1, \dots, x_n)$.

Therefore, we can build a start system of the same size and whose polynomials have the same degrees, but whose solutions are easy to find, and thus can be used as starting points for the method.

For instance, the system $G = (g_1, \dots, g_n)$, where

$$g_i(x_1, \dots, x_n) = x_i^{d_i} - 1,$$

is such a system, since its zero locus is obtained by combining the d_i -th roots of unity in each variable, which are exactly D points:

$$\left\{ \left(e^{\frac{k_1}{d_1} 2\pi i}, \dots, e^{\frac{k_n}{d_n} 2\pi i} \right), \text{ for } 0 \leq k_i \leq d_i - 1 \text{ and } i = 1, \dots, n \right\}.$$

2.1 Choosing the homotopy

The deformation between the original system and the start system is a *homotopy*, for instance the convex combination of F and G

$$H(x, t) = (1 - t)F(x) + tG(x), \tag{1}$$

where $x := (x_1, \dots, x_n)$ and $t \in [0, 1]$. This is such that the roots of $H(x, 0) = G(x)$ are known, and the roots of $H(x, 1) = F(x)$ are the solutions of the original system (the reason why we place the start system at $t = 0$ and the original system at $t = 1$ is that we need higher numerical precision for the solutions of the original system, and there are more floating point numbers near to $t = 0$; see [1], p. 33). Therefore, we can implicitly define a curve $z(t)$ in \mathbb{C}^n by the equation

$$H(z(t), t) = 0, \tag{2}$$

so that in order to approximate the roots of F it is enough to numerically track $z(t)$.

To do so, we derive the expression (2) with respect to t , and get the *Dauidenko Differential Equation*

$$\frac{\partial H}{\partial z} \frac{dz}{dt} + \frac{\partial H}{\partial t} = 0,$$

where $\frac{\partial H}{\partial z}$ is the Jacobian matrix of H with respect to z :

$$\frac{\partial H}{\partial z} = \begin{pmatrix} \frac{\partial H_1}{\partial z_1} & \cdots & \frac{\partial H_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial H_n}{\partial z_1} & \cdots & \frac{\partial H_n}{\partial z_n} \end{pmatrix}.$$

This can be rewritten as

$$\dot{z} = -\frac{\partial H^{-1}}{\partial z} \frac{\partial H}{\partial t}. \quad (3)$$

This is a system of n first-order differential equations, which can be solved numerically for $z(t)$ as an initial value problem, which is called *path tracking*.

2.1.1 Gamma trick

While (1) is a fine choice of a homotopy, it's not what it's called a *good homotopy*: in order to ensure that the solution paths $z(t)$ for different roots

- have no singularities, i.e. never cross each other for $t > 0$ (at $t = 0$ F could have singular solutions), and
- don't go to infinity for $t \rightarrow 0$ (as F could have a solution at infinity),

we can employ the *Gamma trick*: this consists in modifying the linear homotopy (1) by substituting the parameter $t \in [0, 1]$ with a complex curve $q(t)$ connecting 0 and 1, such as

$$q(t) = \frac{\gamma t}{\gamma t + (1 - t)},$$

where $\gamma \in (0, 1)$ is a random complex parameter.

This is a "probability one" procedure, i.e. for any particular system we can choose γ outside of a finite amount of rays through the origin to ensure that we get a good homotopy, basically because of the finiteness of the branch locus of the homotopy. After substituting, we have

$$H(x, t) = \frac{(1 - t)}{\gamma t + (1 - t)} F(x) + \frac{\gamma t}{\gamma t + (1 - t)} G(x),$$

and by clearing denominators, we get our final choice of homotopy:

$$H(x, t) = (1 - t)F(x) + \gamma t G(x). \quad (4)$$

2.2 Tracking down the roots

We then need to track down individual roots, following the solution paths from a root z_0 of the start system by solving the initial value problem associated to the Dauidenko differential equation (3) with starting value z_0 and t ranging from 1 to 0.

This will be done numerically, by using a first-order predictor-corrector tracking method, whose typical iteration goes like this:

- **Predictor:** we first apply Euler's method to get an approximation \tilde{z}_i of the next value of the solution path;
- **Corrector:** we then use Newton's method to correct \tilde{z}_i using equation (2), so that it becomes a good approximation z_i of the next value of the solution path.

In the following sections, we go into more detail on each of these steps.

2.2.1 Predictor: Euler's method

Recall that Euler's method consists in approximating the solution of the initial value problem associated to a system of first-order ordinary differential equations

$$\begin{cases} \dot{z} = f(z, t) \\ z(t_0) = z_0 \end{cases}$$

by the sequence of points $(z_i)_{i \in \mathbb{N}}$ defined by the recurrence relation

$$z_{i+1} = z_i + h \cdot f(z_i, t_i),$$

where h is the step size. In the case of the Davidenko equation (3), we have

$$f(z, t) = - \left(\frac{\partial H}{\partial z}(z, t) \right)^{-1} \frac{\partial H}{\partial t}(z, t)$$

and $t_0 = 1$, since we are tracking from 1 to 0. For the same reason, we set

$$t_{i+1} = t_i - h.$$

2.2.2 Corrector: Newton's method

Since we want to solve

$$H(z, t) = 0,$$

we can use Newton's method to improve the approximation \tilde{z}_i obtained by Euler's method. This is done by moving towards the root of the tangent line of H at the current approximation, or in other words through the iteration

$$z_{i+1} = z_i - \left(\frac{\partial H}{\partial z}(z_i, t_{i+1}) \right)^{-1} H(z_i, t_{i+1}),$$

where this time $z_0 = \tilde{z}_i$, with \tilde{z}_i and t_{i+1} obtained from the i -th Euler step.

Usually, only a few steps of Newton's method are needed; we chose a fixed number of 5 iterations. At which point, we use the final value of the Newton iteration as the starting value for the next Euler step.

2.2.3 Adaptive step size

In order to improve the efficiency of the method, we will use an adaptive step size, which is based on the norm of the residual of Newton's iteration. If the desired accuracy is not reached (say, when the norm of $H(z_i, t_i)$ is bigger than 10^{-8}), then we halve the step size; if instead we have 5 "successful" iterations in a row, we double the step size.

3 Testing the method

To test the method and its scalability, we first launched it on a single-threaded machine, then one a multi-threaded one, and finally parallelized it on a Cluster, whose specifications can be found in the [Hardware](#) section. The latter was done by using the Julia package *Distributed.jl* to parallelize the tracking of the roots on separate nodes, and the `SlurmClusterManager` package, which allows to run Julia code using the Slurm workload manager.

In order to scale the method to larger systems, we also implemented a random polynomial generator, which can be found in [random-poly.jl](#); this was used to create the systems used to evaluate the performance of the parallel implementation.

For sake of visualization, a set of smaller tests was run, in addition to the parallel ones, on a single-threaded machine and a multi-threaded one (using the `@threads` macro from the *Threads.jl* package on the root tracking for loop in the file [solve.jl](#)); however the multi-threaded runs didn't improve the performance on these smaller systems, as the overhead of the multi-threading was too big compared to the actual computation time.

...perhaps because of our choice of predictor-corrector which could be unsuitable for larger systems.

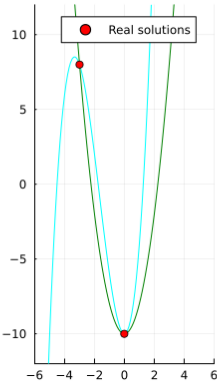
The Julia implementation for the tests described above can be found in [Appendix B](#).

4 Appendix A: Results

4.1 Single- vs Multi-threaded

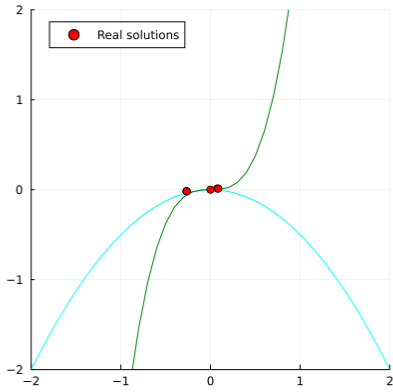
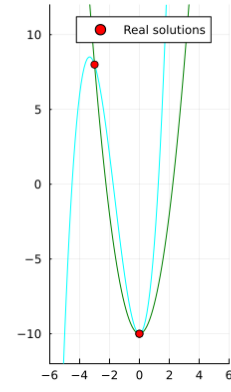
Here are the plots for the solutions of four different 2x2 systems for the single-threaded and multi-threaded cases, with the corresponding systems and the real solutions shown in red.

Single-threaded

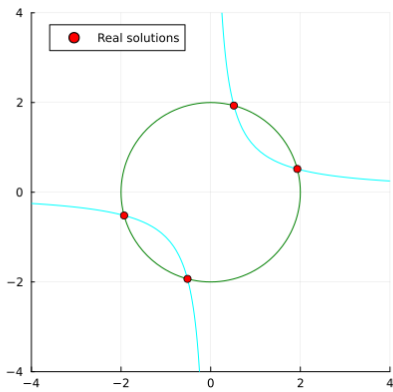
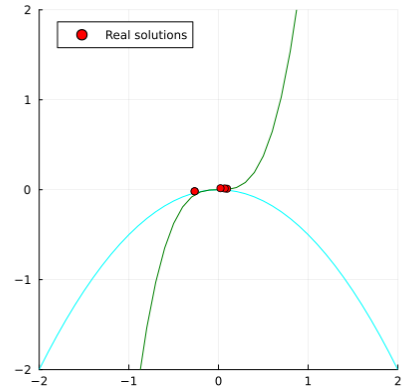


$$\begin{cases} x^3 + 5x^2 - y - 1 \\ 2x^2 - y - 1 \end{cases}$$

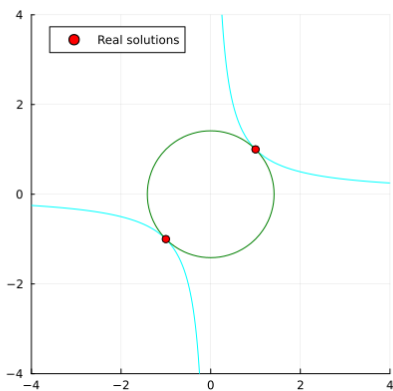
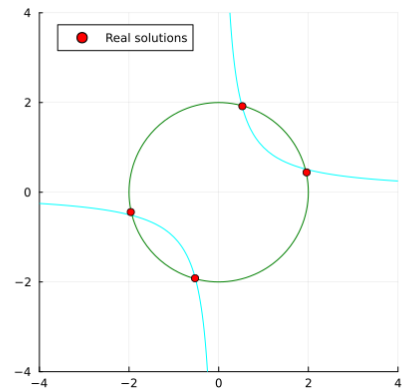
Multithreaded



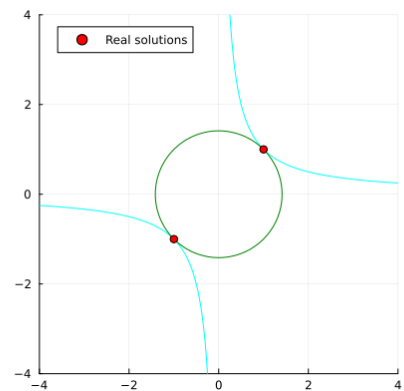
$$\begin{cases} x^2 + 2y \\ y - 3x^3 \end{cases}$$



$$\begin{cases} x^2 + y^2 - 4 \\ xy - 1 \end{cases}$$



$$\begin{cases} x^2 + y^2 - 2 \\ xy - 1 \end{cases}$$



4.2 Parallelization

Below are the plotted residual norms for the solutions of a randomly generated 3x3 system for the parallelized runs, compared with single-threaded runs for the same systems (the latter were run on a single node of the cluster):

The running times for the parallel runs are the following:

5 Appendix B: Implementation

5.1 Julia code

Listing 1: solve.jl

```
1 # External deps
2 using LinearAlgebra
3 using TypedPolynomials
4 using Distributed, SlurmClusterManager
5 slurm_manager = SlurmManager()
6 addprocs(slurm_manager)
7
8 # Local deps
9 include("random-poly.jl")
10 include("plot.jl")
11 using .RandomPoly
12 using .Plot
13 @everywhere begin
14     include("start-system.jl")
15     include("homotopy.jl")
16     include("euler-newton.jl")
17     include("adapt-step.jl")
18 end
19 # Macros defined in an @everywhere block aren't available inside it
20 @everywhere begin
21     using .StartSystem
22     using .Homotopy
23     using .EulerNewton
24     using .AdaptStep
25 end
26
27 @everywhere function compute_root(H, r, maxsteps=200)
28     t = 1.0
29     step_size = 0.001
30     x0 = r
31     m = 0
32     steps = 0
33
34     while t > 0 && steps < maxsteps
35         x0 = en_step(H, x0, t, step_size)
36         (m, step_size) = adapt_step(H, x0, t, step_size, m)
37         t -= step_size
38         steps += 1
39     end
40     return (x0, steps)
41 end
42
43 # Main homotopy continuation loop
44 function solve(F, G, roots)
45     H = homotopy(F, G)
46
47     result = Array{Future}(undef, length(roots))
48     for i in eachindex(roots)
49         result[i] = @spawnat :any compute_root(H, roots[i])
50     end
51
52     sols = Array{ComplexF64,2}(undef, length(roots), length(F))
53     steps = Array{Int64}(undef, length(roots))
```



```

54   for i in eachindex(roots)
55       (solution, step_array) = fetch(result[i])
56       sols[i, :] = solution
57       steps[i] = step_array
58   end
59
60   return (sols, steps)
61 end
62
63 # @polyvar x y
64 # C = [x^3 - y + 5x^2 - 10, 2x^2 - y - 10]
65 # Q = [x^2 + 2y, y - 3x^3]
66 # F = [x*y - 1, x^2 + y^2 - 4]
67 # T = [x*y - 1, x^2 + y^2 - 2]
68
69 R = random_system(5, 5)
70 println("System: ", R)
71 (G, roots)=start_system(R)
72 println("Number of roots: ", length(roots))
73
74 # Parallel execution
75 println("PARALLEL")
76 @time begin
77     (sol, steps) = solve(R, G, roots)
78 end
79 println("Number of steps: ", steps)
80 # converting sR to array of arrays instead of a matrix
81 sol = [sol[i, :] for i in 1:length(sol[:, 1])]
82 sol = filter(u -> imag(u[1]) < 0.1 && imag(u[2]) < 0.1, sol)
83 sol = map(u -> real(u), sol)
84 vars = variables(R)
85 println("Solutions: ", sol)
86 println("Norms (lower = better): ", [norm([f(vars => s) for f in R]) for s in
87     sol])
88
89 # Single execution
90 println("SINGLE")
91 wait(rmprocs(workers()))
92 @time begin
93     (sol, steps) = solve(R, G, roots)
94 end
95 println("Number of steps: ", steps)
96 # converting sR to array of arrays instead of a matrix
97 sol = [sol[i, :] for i in 1:length(sol[:, 1])]
98 sol = filter(u -> imag(u[1]) < 0.1 && imag(u[2]) < 0.1, sol)
99 sol = map(u -> real(u), sol)
100 vars = variables(R)
101 println("Solutions: ", sol)
102 println("Norms (lower = better): ", [norm([f(vars => s) for f in R]) for s in
103     sol])
104
105 # See https://github.com/kleinhenz/SlurmClusterManager.jl/issues/11
106 finalize(slurm_manager)
107
108 # Plotting the system and the real solutions
109 # ENV["GKSwstype"] = "nul"
110 # plot_real(sC, C, 6, 12, "1")
111 # plot_real(sQ, Q, 2, 2, "2")
112 # plot_real(sF, F, 4, 4, "3")
113 # plot_real(sT, T, 4, 4, "4")
114 # plot_real(sol, R, 5, 5, "random")

```

Listing 2: start-system.jl

```

1 module StartSystem
2     using TypedPolynomials
3
4     export start_system
5
6     # Define start system based on total degree

```

```

7 function start_system(F)
8   degrees = [maxdegree(p) for p in F]
9   G = [x_i^d - 1 for (d, x_i) in zip(degrees, variables(F))]
10  r = [[exp(2im*pi/d)^k for k=0:d-1] for d in degrees]
11  roots = vec([collect(root) for root in collect(Iterators.product(r...))])
12  return (G, roots)
13 end
14 end

```

Listing 3: homotopy.jl

```

1 module Homotopy
2   export homotopy
3
4   # Define a straight-line homotopy between the two systems
5   function homotopy(F, G)
6     γ = cis(2π * rand())
7     function H(t)
8       return [(1 - t) * f + γ * t * g for (f, g) in zip(F, G)]
9     end
10    return H
11  end
12 end

```

Listing 4: homogenize.jl

```

1 module Homogenize
2   using TypedPolynomials
3
4   export homogenize, homogenized_start_system
5
6   function homogenize(F)
7     @polyvar h
8     return [sum([h^(maxdegree(p)-maxdegree(t))*t for t in p.terms] for p in F
9   ]
10  end
11  function homogenized_start_system(F)
12    degrees = [maxdegree(p) for p in F]
13    @polyvar h
14    G = [x_i^d - h^d for (d, x_i) in zip(degrees, variables(F))]
15    r = [[exp(2im*pi/d)^k for k=0:d-1] for d in degrees]
16    roots = vec([vcat(collect(root), 1) for root in collect(Iterators.product(r
17    ...))])
18    return (G, roots)
19  end

```

Listing 5: euler-newton.jl

```

1 module EulerNewton
2   using LinearAlgebra
3   using TypedPolynomials
4
5   export en_step
6
7   # Euler-Newton predictor-corrector
8   function en_step(H, x, t, step_size)
9
10    # Predictor step
11    vars = variables(H(t))
12    # Jacobian of H evaluated at (x,t)
13    JH = [jh(vars=>x) for jh in differentiate(H(t), vars)]

```

```

14 #  $\partial H/\partial t$  is the same as  $\nabla G - F = H(1) - H(0)$  for our choice of homotopy
15  $\Delta x = JH \setminus -[gg(\text{vars} \Rightarrow x)$  for  $gg$  in  $H(1) - H(0)$ ]
16  $xh = x + \Delta x * \text{step\_size}$ 
17
18 # Corrector step
19  $JHh = \text{differentiate}(H(\text{t-step\_size}), \text{vars})$ 
20 for  $_$  in 1:5
21    $JH = [jh(\text{vars} \Rightarrow xh)$  for  $jh$  in  $JHh]$ 
22    $\Delta x = JH \setminus -[h(\text{vars} \Rightarrow xh)$  for  $h$  in  $H(\text{t-step\_size})]$ 
23    $xh = xh + \Delta x$ 
24 end
25
26 return  $xh$ 
27 end
28 end

```

Listing 6: adapt-step.jl

```

1 module AdaptStep
2   using LinearAlgebra
3   using TypedPolynomials
4
5   export adapt_step
6
7   # Adaptive step size
8   function adapt_step(H, x, t, step, m)
9      $\Delta = \text{norm}([h(\text{variables}(H(\text{t})) \Rightarrow x)$  for  $h$  in  $H(\text{t-step})])$ 
10    if  $\Delta > 1e-8$ 
11      step =  $0.5 * \text{step}$ 
12      m = 0
13    else
14      m += 1
15      if (m == 4) && (step < 0.05)
16        step =  $2 * \text{step}$ 
17        m = 0
18      end
19    end
20
21    return (m, step)
22  end
23 end

```

Listing 7: random-poly.jl

```

1 module RandomPoly
2   export random_system
3
4   using TypedPolynomials
5   using Random
6   using Distributions
7
8   # Random polynomial of degree n in m variables
9   function random_poly(n, m)
10     $x = [\text{TypedPolynomials.Variable}\{\text{Symbol}("x[\$i]")\}() \text{ for } i \text{ in } 1:m]$ 
11
12    monomial_powers = collect(Iterators.product([0:n for  $_$  in 1:m]...))
13    monomials = [ $\text{prod}(x.^i)$  for  $i$  in monomial_powers if  $\text{sum}(i) \leq n$  &&  $\text{sum}(i) !$ 
14      = 0]
15
16    return sum(map( $m \rightarrow \text{rand}(\text{Normal}()) * m$ , monomials))
17  end
18
19  # Generate a system of m random polynomials in m variables
20  # of degree  $d_i$  randomly chosen between 1 and max_degree
21  function random_system(m, max_degree)
22     $d = \text{rand}(1:\text{max\_degree}, m)$ 

```

```

22     random_polys = [random_poly(d[i], m) for i in 1:m]
23
24     return random_polys
25 end
26 end

```

Listing 8: plot.jl

```

1 module Plot
2     using Plots, TypedPolynomials
3
4     export plot_real
5
6     function plot_real(solutions, F, h, v, name)
7         plot(xlim = (-h, h), ylim = (-v, v), aspect_ratio = :equal)
8         contour!(-h:0.1:h, -v:0.1:v, (x,y)->F[1](variables(F)->[x,y]), levels=[0],
9                 cbar=false, color=:cyan)
10        contour!(-h:0.1:h, -v:0.1:v, (x,y)->F[2](variables(F)->[x,y]), levels=[0],
11                cbar=false, color=:green)
12        scatter!([real(sol[1]) for sol in solutions], [real(sol[2]) for sol in
13                solutions], color = "red", label = "Real solutions")
14
15        png(joinpath("./plots", "solutions" * name))
16    end
17 end

```

5.2 Hardware

For the single-threaded runs, the code was executed on a laptop with an Intel Core i7-3520M CPU @ 3.60GHz and 6 GB of RAM.

The multithreaded runs were tested on a desktop with an AMD FX-8350 CPU @ 4.00GHz with 4 cores and 8 threads, and 12 GB of RAM.

Finally, the parallel computations were run on a cluster with 20 nodes, each having a CPU @ 1.008GHz with 4 Performance cores, 2 efficiency cores and 4 GB of RAM.

References

- [1] Bates, Daniel J. *Numerically solving polynomial systems with Bertini*. SIAM, Society for Industrial Applied Mathematics, 2013.
- [2] <https://docs.julialang.org/en/v1/stdlib/Distributed>