Def $\mathbb{C} = (\mathbb{R}^2, +_{\mathbb{C}}, \cdot_{\mathbb{C}})$ con

① $(a,b)+(c,d) = (a+c, b+d)$

② $(a,b)\cdot(c,d) = (ac-bd, bc+ad)$

Oss $\mathbb{C}$ e' un campo

Notazione $1=(1,0)$ $i=(0,1)$, notiamo che

$i^2 = (0,1)\cdot(0,1) = (-1,0) = -1.$

Con questa notazione $z\in\mathbb{C}$, $z=(a,b)$ si può scrivere come $z = a\cdot 1 + ib = a+ib$

Notazione Se $z=a+bi$, allora $Re(z)=a$ e $Im(z)=b$.

Def [Coniugio] $\bar{\cdot}: \mathbb{C}\to\mathbb{C}$
$z=a+ib \to a-ib = \bar{z}$

Proprieta' ① $Re(\bar{z}) = Re(z)$, $Im(\bar{z}) = -Im(z)$

② $\boxed{\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2}$ infatti $Re(\overline{z_1+z_2}) = Re(z_1+z_2) = Re(z_1)+Re(z_2)$

$Im(\overline{z_1+z_2}) = -Im(z_1+z_2) = -Im(z_1) - Im(z_2)$

③ $\boxed{\bar{z}_1\bar{z}_2 = \overline{z_1 z_2}}$ $\quad (\overset{z_1}{\overbrace{a,b}})\cdot(\overset{z_2}{\overbrace{c,d}}) = (ac-bd, ad+bc)$

$(Re(z_1), Im(z_1))(Re(z_2), Im(z_2)) = (Re(z_1)Re(z_2) - Im(z_1)Im(z_2), Re(z_1)Im(z_2)+Re(z_2)Im(z_1))$

$Re(\bar{z}_1\bar{z}_2) = Re(\bar{z}_1)Re(\bar{z}_2) - Im(\bar{z}_1)Im(\bar{z}_2) =$

$= Re(z_1)Re(z_2) - Im(z_1)Im(z_2)$ (*)

$Re(\overline{z_1z_2}) = Re(z_1z_2) =$ (*)

④ $z\in\mathbb{R}\times\{0\}$ allora $\bar{z}=z$ $\qquad$ ⑥ $\overline{(\bar{z})} = z$

⑤ $z\in\{0\}\times\mathbb{R}$ allora $\bar{z}=-z$ $\qquad$ ⑦ $\overline{(z^{-1})} = (\bar{z})^{-1}$

Def $|\cdot|: \mathbb{C}\to[0,+\infty)$
$a+ib = z \to |z| = \sqrt{a^2+b^2}$

Oss $\mathbb{C}$ era un campo? $z\in\mathbb{C}\setminus\{0\}$, allora $z^{-1} = \frac{\bar{z}}{|z|^2}$.
Infatti $zz^{-1} = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$.

Oss $|z|^2 = z\bar{z}$, infatti $(a+ib)(a-ib) = a^2+b^2+i(ab-ab) = a^2+b^2$

Oss $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2i}$

Def $e^{i\theta} := \cos(\theta) + i\sin(\theta)$, $\theta\in\mathbb{R}$

Proprieta' ① $(e^{i\theta_1})(e^{i\theta_2}) = e^{i(\theta_1+\theta_2)}$, infatti $(\cos(\theta_1)+i\sin(\theta_1))(\cos(\theta_2)+i\sin(\theta_2)) =$

$= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\ldots) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

② $(e^{i\theta})^n = e^{in\theta}$, $n\in\mathbb{N}$

FORMULA DI DE-MOIVRE

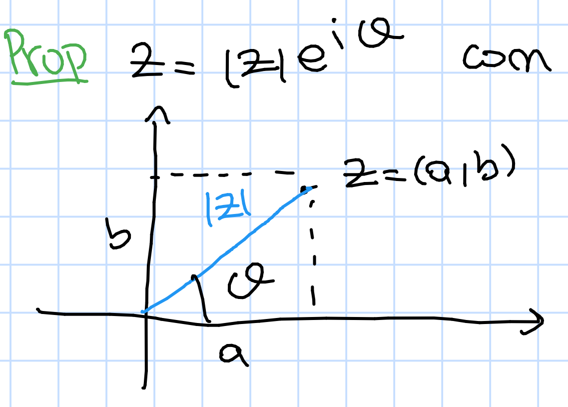
$(\cos(\theta)+i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$

$\boxed{n=1}$

$\boxed{n\to n+1}$ $\quad (\cos(\theta)+i\sin(\theta))^{n+1} = (\cos\theta+i\sin\theta)^n(\cos\theta+i\sin\theta) =$

$= (\cos(n\theta)+i\sin(n\theta))(\cos\theta+i\sin\theta) = \cos((n+1)\theta) + i\sin((n+1)\theta).$

Prop $z = |z|e^{i\theta}$ con $\theta\in[0,2\pi)$ [COORDINATE POLARI]



$a = |z|\cos\theta \qquad b = |z|\sin\theta$

$z = a+ib = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$

Esercizio $\boxed{z^n = (|z|e^{i\theta})^n = |z|^n e^{in\theta}}$

$z = |z|e^{i\theta} \leadsto \frac{z}{|z|} = e^{i\theta} = \cos\theta + i\sin\theta$

$z = 1+i \leadsto |z| = \sqrt{1^2+1^2} = \sqrt{2} \leadsto \frac{z}{|z|} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})$

$(1+i)^{120} = (|z|e^{i\pi/4})^{120} = |z|^{120}e^{i120\pi/4} = 2^{60}e^{i30\pi} = 2^{60}$

Oss $p(x)\in\mathbb{R}[x]$, $\deg p$ e' dispari. Allora $\exists x_0\in\mathbb{R}$ tale che $p(x_0)=0$.

Infatti $p(x) = a_n x^n + \ldots + a_0$ con n dispari, $a_n > 0$.

$\lim_{x\to+\infty} p(x) = +\infty$, $\lim_{x\to-\infty} p(x) = -\infty$, p continua $\Rightarrow \exists x_0\in\mathbb{R}$ $p(x_0)=0$.

Oss $p(x)\in\mathbb{R}[x]$, $z\in\mathbb{C}$ radice di $p \Rightarrow \bar{z}\in\mathbb{C}$ radice di p.

Infatti $p(\bar{z}) = a_n(\bar{z})^n + \ldots + a_0 \underset{\bar{z}\cdots\bar{z} = \overline{z^n}}{=} a_n\overline{z^n} + \ldots + a_0 \underset{\bar{a}_n = a_n \wedge \overline{z_1z_2} = \bar{z}_1\bar{z}_2}{=} \overline{a_nz^n} + \ldots + \overline{a_0} =$

$\underset{\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2}{=} \overline{a_nz^n + \ldots + a_0} = \overline{p(z)} = \bar{0} = 0$

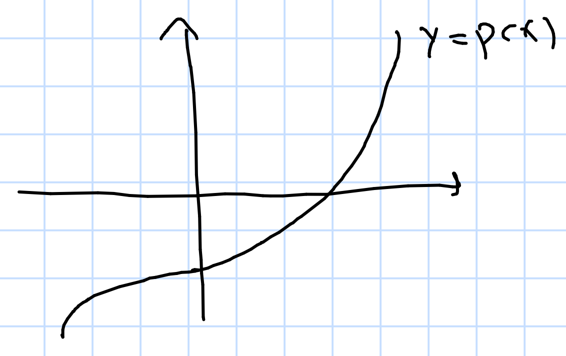
Esercizio Radici di $p(z) = z^3 - 8$.

Soluzione $p\in\mathbb{R}[x]$, $\deg p$ e' dispari. Allora $\exists x_0\in\mathbb{R}$ $p(x_0)=0$.

Inoltre se $z\in\mathbb{C}$ e' radice, allora $\bar{z}\in\mathbb{C}$ radice. Dunque

$R(p) = \{\underset{\mathbb{R}}{x_0}, \underset{\mathbb{C}\setminus\mathbb{R}}{z_0}, \underset{\mathbb{C}\setminus\mathbb{R}}{\bar{z}_0}\}$ oppure $R(p) = \{\underset{\mathbb{R}}{x_0}, \underset{\mathbb{R}}{x_2}, \underset{\mathbb{R}}{x_2}\}$

Studiamo $p: \mathbb{R}\to\mathbb{R}$. Notiamo che $p'(x) = 3x^2$, ovvero p ha un solo punto critico $(x=0)$ che e' un punto di flesso.



$\}\to$ ha una sola radice reale

Sia $z = |z|e^{i\theta}$ con $\theta\in[0,2\pi)$. Allora

$z^3 = |z|^3e^{i3\theta} = 8$ SSE $|z|^3(\cos(3\theta)+i\sin(3\theta)) = 8$ SSE

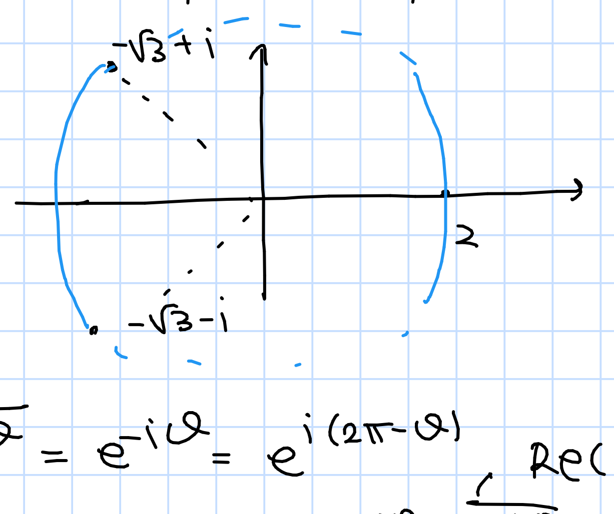
SSE $\underbrace{\sin(3\theta)=0}_{A} \wedge \underbrace{|z|^3\cos(3\theta) = 8}_{B}$

Ⓐ $3\theta = 2\pi k \leadsto \theta = \frac{2\pi}{3}k \leadsto \theta\in\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ $\quad (\theta\in[0,2\pi))$

Ⓑ $|z|^3\cos(3\theta) = 8$ SSE $|z|^3\cos(0, 2\pi, 4\pi) = 8$ SSE $|z|^3 = 8$ SSE $|z| = 2$.

Quindi $z\in\{2, 2(\cos(\frac{2\pi}{3}) + i\sin(\frac{2}{3}\pi)), 2(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}))\} =$

$= \{2, -\sqrt{3}+i, -\sqrt{3}-i\}$



Oss $\overline{e^{i\theta}} = e^{-i\theta} = e^{i(2\pi-\theta)}$, $Re(e^{i\theta}) = \cos(\theta)$, $Im(e^{i\theta}) = \sin(\theta)$

$\sin(\theta) = Im(e^{i\theta}) = \frac{e^{i\theta} - \overline{e^{i\theta}}}{2i} = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos(\theta) = Re(e^{i\theta}) = \frac{e^{i\theta} + \overline{e^{i\theta}}}{2} = \frac{e^{i\theta} + e^{-i\theta}}{2}$

Esercizio $\sin^2(x) = 1 - \cos^2(x) \underset{\cos(2x) = \cos^2(x) - \sin^2(x)}{=} 1 - \cos(2x) - \sin^2(x)$

$\leadsto \sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$

$\sin^2(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 = \frac{e^{i2x} + e^{-i2x} - 2}{-4} = \frac{1}{2} - \frac{\cos(2x)}{2}$

$\sin^3(x) = \frac{(e^{ix} - e^{-ix})^3}{(2i)^3} = \frac{e^{i3x} - 3e^{i2x}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-i3x}}{-8i} =$

$= -\frac{1}{4}\left(\frac{e^{i3x} - e^{-i3x}}{2i}\right) + \frac{3}{4}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) = -\frac{\sin(3x)}{4} + \frac{3}{4}\sin(x)$