Esercitazione 15

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DUALE.

$f: V \to \mathbb{K}$ $\quad f \in V^*$

$\bullet \, \mathrm{Im} f = \begin{cases} \{0\} \iff f = 0 \\ \mathbb{K} \end{cases}$ $\qquad \bullet \, \dim \mathrm{Ker} f = \begin{cases} n \iff f = 0 & \mathrm{Ker} f = V \\ n-1 & \mathrm{Ker} f = \text{iperpiano} \end{cases}$

IPERPIANO

$W \subset V$ iperpiano $\quad U \subset V$ ssp.

$U + W = \begin{cases} V & \longrightarrow \dim U + W = n \\ W \iff W \supset U & \dim U + W = n - 1 \end{cases}$

$\dim U \cap W = \dim U + \underset{\substack{\shortparallel \\ n-1}}{\dim W} - \dim(U+W) = \begin{cases} \underset{n}{\dim V} \text{ se } U \subset W \\ \underset{n-1}{\dim V - 1} \end{cases}$

$(U+W)$: n, $n-1$

CRITERIO per stabilire se un funzionale è multiplo di un altro

$f, g \in V^*$, $\exists \lambda \in \mathbb{K}$ t.c. $f = \lambda g \iff \mathrm{Ker}\, g \subset \mathrm{Ker} f$

dim

$\Rightarrow$ $\underline{v} \in \mathrm{Ker}\, g$, $f(\underline{v}) = (\lambda g)(\underline{v}) = \lambda g(\underline{v}) = \lambda \underline{0} = \underline{0} \Rightarrow \underline{v} \in \mathrm{Ker} f$

$\Leftarrow$ se $\dim \mathrm{Ker} f = n - 1 \Rightarrow \dim \mathrm{Ker}\, g = n-1$ perché $\dim \mathrm{Ker}\, g = n, n-1$
ma per hp $\mathrm{Ker}\, g \subset \mathrm{Ker} f \Rightarrow \dim \mathrm{Ker}\, g \leq \dim \mathrm{Ker} f$.
$\Rightarrow \mathrm{Ker}\, g = \mathrm{Ker} f$.

Sia $\underline{v} \notin \mathrm{Ker}\, g = \mathrm{Ker} f \Rightarrow f(\underline{v}), g(\underline{v}) \neq 0$

pongo $\lambda = \dfrac{f(\underline{v})}{g(\underline{v})}$ verifico che $f = \lambda g$.

$V = \mathrm{Ker} f \oplus \mathrm{Span}(\underline{v})$ perché $\underline{v} \notin \mathrm{Ker} f$.

Sia $\{\underline{u_1}, \dots, \underline{u_{n-1}}\}$ base di $\mathrm{Ker} f = \mathrm{Ker}\, g$.
completo.

sia $\{ \underline{u_1}, \dots, u_{n-1} \}$ base di $\operatorname{Ker} f = \operatorname{Ker} g$.

completo.

$\{ \underline{u_1}, \dots, u_{n-1}, \underline{v} \}$ base B di V

verifico $f = \lambda g$ su B.

$$f(u_j) = 0 = \lambda g(u_j) \qquad \forall j = 1, \dots, n-1.$$

$u_j \in \operatorname{Ker} f$. $\quad u_j \in \operatorname{Ker} g$.

$$f(\underline{v}) = \lambda g(\underline{v})$$

Quindi $\rightarrow$ $f, g \in V^*$ sono lin. indip $\Longleftrightarrow$ $\operatorname{Ker} f, \operatorname{Ker} g \neq V$

e

equi $\quad \operatorname{Ker} f \neq \operatorname{Ker} g$

$\Longleftrightarrow \dim(\operatorname{Ker} f \cap \operatorname{Ker} g) = n-2$

Base duale

$\underline{e_1}^T = (1\ 0 \dots 0) \quad \dots \quad \underline{e_n}^T = (0 \dots 0\ 1)$ sono una base di $M(1, n, \mathbb{K})$

$$f_i = \left([\ ]^B_{can} \right)^{-1} (\underline{e_i}^T) \qquad \forall i = 1, \dots, n$$

i f_i sono una base B^* di V^* base duale di B.

$$f_j(\underline{v_i}) = e_j^T e_i = (0 \dots \underset{j}{1} \dots 0) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} i = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

$\underline{v} \in V \qquad [v]_B = \begin{pmatrix} f_1(\underline{v}) \\ f_2(\underline{v}) \\ \vdots \\ f_n(\underline{v}) \end{pmatrix}$

$v = a_1 \underline{v_1} + \dots + a_n \underline{v_n}$

$$f_j(\underline{v}) = f_j(a_j \underline{v_j}) = a_j f_j(\underline{v_j}) = a_j$$

$f \in V^* \qquad [f]_{B^*} = \begin{pmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{pmatrix}$

$$f(v_j) = b_j f_j(v_j) = b_j$$

$$f = b_1 f_1 + \dots + b_n f_n.$$

Esercizio.

Data una base B di V trovare B^*

$\underline{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ base B di $\mathbb{R}^3$

$B^* = \{f_1, f_2, f_3\}$ $f_1, f_2, f_3 \in (\mathbb{R}^3)^*$

$f_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = (a\ b\ c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$ con $a, b, c \in \mathbb{R}$.

$f_1(\underline{v_1}) = 1$ $f_2(\underline{v_1}) = 0$ $f_3(\underline{v_1}) = 0$

$f_1(\underline{v_2}) = 0$ $f_2(\underline{v_2}) = 1$ $f_3(\underline{v_2}) = 0$

$f_1(\underline{v_3}) = 0$ $f_2(\underline{v_3}) = 0$ $f_3(\underline{v_3}) = 1$

$\downarrow$

$$\begin{aligned} f_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} &= \\ f_1(\underline{v_2}) &= \\ f_1(\underline{v_3}) &= \end{aligned} \begin{cases} a + c = 1 \\ 2a + 2b + c = 0 \\ -2a - 2b + c = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $c = 0$ $a = 1$ $b = -1$

$\Rightarrow f_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x - y$

$f_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = -\frac{1}{2}x + \frac{3}{4}y + \frac{1}{2}z$ $f_3\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = -\frac{1}{2}x + \frac{1}{4}y + \frac{1}{2}z$.

$\underline{v} \in \mathbb{R}^3$ $\underline{v} = f_1(\underline{v})(\underline{v_1}) + f_2(\underline{v})(\underline{v_2}) + f_3(\underline{v})(\underline{v_3})$

$f \in (\mathbb{R}^3)^*$ $f = f(\underline{v_1})f_1 + f(\underline{v_2})f_2 + f(\underline{v_3})f_3$

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]_B = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]_B = \begin{pmatrix} -1 \\ 3/4 \\ 1/4 \end{pmatrix} \qquad \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]_B = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -\frac{1}{2} & 3/4 & 1/2 \\ -\frac{1}{2} & 1/4 & 1/2 \end{pmatrix} = \mathfrak{M}_{B}^{Can}(id_{\mathbb{R}^3})$$

$$[x]_{B^*} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \qquad [y]_{B^*} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \qquad [z]_{B^*} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathfrak{M}_{B^*}^{Can^*}(id_{\mathbb{R}^{3*}}) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ -2 & -2 & 1 \end{pmatrix}$$

Matrici righe e funzionali

$A \in GL(n, \mathbb{K})$ $\quad (A^1, \dots, A^n)$ sono una base B di $\mathbb{K}^n$.

Le righe di A^{-1} $\quad A^{-1} = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}$ $\quad B_i \in \mathcal{M}(1, n, \mathbb{K}) \Rightarrow B_i \in (\mathbb{K}^n)^*$

$A A^{-1} = I \quad B_j A^i = \begin{cases} 1 & \text{se } j = i \\ 0 & \text{se } i \neq j \end{cases} \Rightarrow B_1, \dots, B_n$ sono la base B^* di $(\mathbb{K}^n)^*$

Penso le matrici righe come funzionali.

Annullatore

$S \subset V$ sottoinsieme

$$\begin{aligned} Ann(S) &= \{ f \in V^* \text{ t.c. } f(\underline{s}) = 0 \ \forall \underline{s} \in S \} \\ &= \{ f \in V^* \text{ t.c. } S \subset \ker f \} \\ &= \{ f \in V^* \text{ t.c. } Span(S) \subset \ker f \} = Ann(Span(S)) \end{aligned}$$

Proprietà

1 $Ann(S) = Ann(Span(S))$

2 $S \subset T \subset V \Rightarrow Ann(S) \supset Ann(T)$ $\rightarrow$ rovescia le inclusioni

2 $S \subset T \subset V \Rightarrow Ann(S) \supset Ann(T)$ $\longrightarrow$ rovescia le inclusioni

3 $Ann(\{\underline{0}\}) = V^*$ $\quad Ann(V) = \{\underline{0}\} \subset V^*$

4 $\dim V = n$ $\quad \dim Ann(S) = n - \dim(Span(S))$

5 $Ann : \{ssp \text{ di } V\} \longmapsto \{ssp \text{ di } V^*\}$

6 $W \subset V$ ssp. $\quad w_1, \dots, w_k$ base di W $\quad \mathcal{B} = \{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ base di V.

$\mathcal{B}^* = \{f_1, \dots, f_n\}$ $\quad Ann(W) = Span(f_{k+1}, \dots, f_n)$

es di prima $\quad W = Span(f_1, f_2) \Rightarrow Ann(W) = Span(f_3)$

$W = Span(f_1) \Rightarrow Ann(W) = Span(f_2, f_3)$

7 $Ann(U+W) = Ann(U) \cap Ann(W)$

$\subset$

$$\left.\begin{array}{l} U \subset U+W \\ W \subset U+W \end{array}\right\} \Rightarrow \begin{array}{l} Ann(U+W) \subset Ann(U) \\ \qquad\qquad\quad\;\; \subset Ann(W) \end{array}$$

$\Rightarrow Ann(U+W) \subset Ann(U) \cap Ann(W)$

$\supset$ $f \in Ann(U) \cap Ann(W)$ $\quad \underline{v} \in U+W$ $\quad \underline{v} = \underline{u} + \underline{w}$.

$f(\underline{v}) = f(u+w) = f(u) + f(w) = 0 + 0 = 0 \Rightarrow f \in Ann(U+W)$

8 $Ann(U \cap W) = Ann(U) + Ann(W)$

$\supset$

$$\begin{array}{l} U \cap W \subset U \\ U \cap W \subset W \end{array} \Rightarrow \begin{array}{l} Ann(U \cap W) \supset Ann(U) \\ \qquad\qquad\quad\;\; \supset Ann(W) \end{array} \Rightarrow$$

$Ann(U \cap W) \supset Ann(U) + Ann(W)$

$\subset$

$\dim(Ann(U) + Ann(W)) = \dim Ann\, U + \dim Ann\, W -$

c

$$\dim(\mathrm{Ann}(U)+\mathrm{Ann}(W)) = \dim \mathrm{Ann}\, U + \dim \mathrm{Ann}\, W -$$
$$- \dim(\mathrm{Ann}\, U \cap \mathrm{Ann}\, W) =$$
$$= n - \dim U + n - \dim W - \dim \mathrm{Ann}(U+W) =$$
$$= 2n - \dim U - \dim W - n + \dim(U+W) =$$
$$= n - \cancel{\dim} U - \cancel{\dim} W + \cancel{\dim} U + \cancel{\dim} W - \dim(U \cap W)$$
$$= n - \dim(U\cap W) = \dim(\mathrm{Ann}(U\cap W))$$

Biduale

$V^{**} = \mathrm{Hom}(V^*, K)$

$\underline{v} \in V \qquad \mathrm{val}_{\underline{v}} : V^* \longrightarrow K \qquad \mathrm{val}_v \in V^{**}$
$\qquad\qquad\qquad\qquad f \longmapsto f(\underline{v})$

$\phi : V \longrightarrow V^{**}$
$\quad \underline{v} \longmapsto \mathrm{val}_{\underline{v}}$

ϕ è iso canonico.

Ann(Ann(W))

$W \subset V$ ssp.

$$\underbrace{\mathrm{Ann}(\underbrace{\mathrm{Ann}(W)}_{\subset V^*})}_{\subset V^{**}} = \underbrace{\phi(W)}_{\subset V^{**}}$$

$\supset \quad f \in \phi(W) \qquad f = \mathrm{val}_w \quad w \in W$

$g \in \mathrm{Ann}(W) \qquad f(g) = \mathrm{val}_w(g) = g(w) = 0 \Rightarrow$

$\Rightarrow f \in \mathrm{Ann}(\mathrm{Ann}(W))$

ug. tra dim.

$\dim \phi(W) = \dim W$

ug. tra dim.

$\dim \phi(W) = \dim W$

$\dim Ann(Ann(W)) = n - \dim(Ann(W)) = n - (n - \dim W)$

$= \dim W$

$$\{ssp\ di\ V\} \xrightarrow{Ann} \{ssp\ di\ V^*\} \xrightarrow{Ann} \{ssp\ di\ V^{**}\}$$

Z ϕ

Chi è Z?

proposizione

$U, W \subset V$ ssp $\quad U = W \iff Ann(U) = Ann(W)$

$\Rightarrow$ ovvia

$\Leftarrow$ $Ann(U) = Ann(W) \Rightarrow Ann(Ann(U)) = Ann(Ann(W))$

$$\underbrace{Ann(Ann(U))}_{\phi(U)} = \underbrace{Ann(Ann(W))}_{\phi(W)}$$

applico $f^{-1} \Rightarrow U = W$.

Proposizione

$U \subset W \iff Ann(U) \supset Ann(W)$

$\Rightarrow$ $U \subset W \Rightarrow Ann(U) \supset Ann(W)$ Ann rovescia le inclusioni

$\Leftarrow$ $\underbrace{Ann(Ann(U))}_{\phi(U)} \subset \underbrace{Ann(Ann(W))}_{\phi(W)}$

ϕ^{-1}

$\Rightarrow U \subset W$.

F trasposta

$f^t: W^* \longrightarrow V^*$

- $\operatorname{Ker} f^T = \{ g \in W^* \text{ t.c. } g \circ f = 0 \} = \{ g \in W^* \mid \operatorname{Ker} g \supset \operatorname{Im} f \} =$
  $= \operatorname{Ann}(\operatorname{Im} f)$

- $\operatorname{Im} f^t \supset \operatorname{Ann}(\operatorname{Ker} f)$
  $=$ se in dim finita.

- B base di V, C base di W $\mathcal{M}^{C^*}_{B^*} = (\mathcal{M}^{B}_{C}(f))^T$.