Esercitazione 6

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Valutazione di un pol. in a.

$Val_a : \mathbb{K}[t] \longrightarrow \mathbb{K}$ $\quad a \in \mathbb{K}$ $\quad Val_a$ è lineare

$p \longrightarrow p(a)$

$\operatorname{Im} Val_a = \mathbb{K}$ cioè Val_a è surgettiva.

$\operatorname{Ker} Val_a = (t-a)$ cioè Val_a non è inj.

$p \in \operatorname{Ker} Val_a \iff p(a) = 0 \iff p = (t-a)p_1 \quad p_1 \in \mathbb{K}[t]$

$(t-a) \equiv$ ideale generato da $t-a$.

Esercizio 1

Sia $p_0 \in \mathbb{K}[t]$ $\quad p_0(a) \neq 0$ cioè $p_0 \notin \operatorname{Ker}(Va(a))$

Allora $\quad \mathbb{K}[t] = \operatorname{Ker}(Val_a) \oplus \operatorname{Span}(p_0)$

Sol

Devo mostrare:

1. $\operatorname{Span}(p_0) \cap \operatorname{Ker}(Val_a) = \{0\}$
2. $\mathbb{K}[t] = \operatorname{Span}(p_0) \oplus \operatorname{Ker}(Val_a)$

1) Sia $p \in \operatorname{Span}(p_0) \cap \operatorname{Ker}(Va(a))$

$$\begin{cases} p \in \operatorname{Span}(p_0) \Rightarrow p = \lambda p_0 & \lambda \in \mathbb{K}. \\ p \in \operatorname{Ker}(Val_a) \Rightarrow p(a) = 0 \end{cases}$$

$\Rightarrow 0 = p(a) = \lambda p_0(a) = 0$ ma $p_0(a) \neq 0$ per hp

$\Rightarrow \lambda = 0 \Rightarrow p = 0 \Rightarrow \operatorname{Span}(p_0) \cap \operatorname{Ker}(Val_a) = \{0\}$

2) Sia $p \in \mathbb{K}[t]$ posso scrivere p come

$p = q + \lambda p_0$ $\quad$ dove $q \in \operatorname{Ker}(Val_a) \Rightarrow q(a) = 0$

$p = q + \lambda p_0$ dove $q \in \mathrm{Ker}(Val_a) \Rightarrow q(a) = 0$

$\lambda p_0 \in \mathrm{Span}(p_0)$

Trovo λ e q $\longrightarrow$ valuto in a

$$p(a) = (q + \lambda p_0)(a) = \underset{\substack{\parallel \\ 0 \\ q \in \mathrm{Ker}(Val_a)}}{q(a)} + \lambda p_0(a) = \lambda p_0(a)$$

$$\Rightarrow \lambda = \frac{p(a)}{p_0(a)} \quad e \ p_0(a) \neq 0.$$

$$q = p - \lambda p_0 = p - \frac{p(a)}{p_0(a)} p_0$$

Verifico che $q \in \mathrm{Ker}\, Val_a$.

$$q(a) = \left(p - \frac{p(a)}{p_0(a)} p_0\right)(a) = p(a) - \frac{p(a)}{\cancel{p_0(a)}} \cancel{p_0}(a) = p(a) - p(a) = 0$$

Esercizio 2

V $\mathbb{K}$ sp. v.

$f: V \to \mathbb{K}$ lineare

$f \neq 0 \Rightarrow f$ surg.

$\underline{v_0} \in V$ e $\underline{v_0} \notin \mathrm{Ker} f$

$V = \mathrm{Ker} f \oplus \mathrm{Span}(\underline{v_0}) \longrightarrow$ Tesi

Sol

1) $\mathrm{Ker} f \cap \mathrm{Span}(v_0) = \{0\}$

Sia $v \in \mathrm{Ker} f \cap \mathrm{Span}(\underline{v_0}) \Rightarrow f(v) = 0$ e $\underline{v} = \lambda \underline{v_0}$

$\Rightarrow 0 = f(\underline{v}) = f(\lambda \underline{v_0}) = \lambda f(\underline{v_0}) = 0$ ma $f(\underline{v_0}) \neq 0$ per hp

$\Rightarrow \lambda = 0 \Rightarrow \underline{v} = 0 \Rightarrow \ker f \cap \operatorname{Span}(\underline{v}) = \{0\}$

2) $V = \ker f \oplus \operatorname{Span}(v_0)$

$\underline{v} \in V \quad \underline{v} = \underline{w} + \lambda \underline{v_0}$ con $\underline{w} \in \ker f$

applico $f \rightarrow f(\underline{v}) = f(\underline{w} + \lambda \underline{v_0}) = f(\underline{w}) + f(\lambda \underline{v_0}) =$

$= \lambda f(\underline{v_0}) \Rightarrow \boxed{\lambda = \frac{f(\underline{v})}{f(\underline{v_0})}}$

$\underline{w} = \underline{v} - \lambda \underline{v_0} = \boxed{\underline{v} - \frac{f(\underline{v})}{f(\underline{v_0})} \underline{v_0} = \underline{w}}$

$\underline{w} \in \ker f \rightarrow 0 = f(\underline{w}) = f\left(\underline{v} - \frac{f(\underline{v})}{f(\underline{v_0})} \underline{v_0}\right) =$

$= f(\underline{v}) - \frac{f(\underline{v})}{\cancel{f(\underline{v_0})}} \cancel{f(\underline{v_0})} = 0.$

Esercizio 3

$M(n, \mathbb{K}) = S_n \oplus A_n$

1) $S_n \cap A_n = \{0\}$

Sia $A \in S_n \cap A_n \Rightarrow A \in S_n \Rightarrow A^T = A$
$A \in A_n \Rightarrow A^T = -A$
$\Rightarrow A = -A \Rightarrow 2A = 0$
se $2 \neq 0$ in $\mathbb{K}$.

2) $M(n, \mathbb{K}) = S_n \oplus A_n$

$\supset$ $S_n \subset M(n, \mathbb{K})$
$A_n \subset M(n, \mathbb{K})$
$\Rightarrow$ dato che $M(n, \mathbb{K})$ è sp. v. è chiuso $+$.
$\Rightarrow S_n + A_n \in M(n, \mathbb{K})$

$An \subset M(n, \mathbb{K})$ $\Rightarrow$ $Sn + An \in M(n, \mathbb{K})$

$\supset$ Sia $A \in M(n, \mathbb{K})$

$A + A^T \in Sn$

$(A + A^T)^T = A^T + (A^T)^T = A^T + A \in Sn.$

$A - A^T \in An$

$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(-A^T + A) = \underbrace{-(A - A^T)}_{\in An.}$

$(A + A^T) + (A - A^T) = 2A$

$\underbrace{\frac{1}{2}(A + A^T)}_{\in Sn.} + \underbrace{\frac{1}{2}(A - A^T)}_{\in An} = A$ ho scritto A come un pezzo $\in Sn$ e uno $\in An$.

esercizio 4

V sp. v su $\mathbb{K}$ $f: V \to V$ line. $f^2 = id$.

$V_1 = \{ v \in V$ t.c. $f(v) = v \}$

$V_{-1} = \{ v \in V$ t.c. $f(v) = -v \}$.

$V = V_1 \oplus V_{-1}$

1) $V_1 \cap V_{-1} = \{0\}$ $\quad$ $2 \neq 0$ in $\mathbb{K}$?

$v \in V_1 \cap V_{-1} \Rightarrow f(v) = v$ e $f(v) = -v \Rightarrow 2v = 0 \Rightarrow v = 0$

2) $V = V_1 \oplus V_{-1}$

$\supset$ $v_1 \in V$, $v_{-1} \in V \Rightarrow v_1 + v_{-1} \in V$

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Sia $v \in V$ $\quad v = \frac{1}{2}(v + f(v)) + \frac{1}{2}(v - f(v)) = v$

$v + f(v) \in V_1 \rightarrow f(v + f(v)) = f(v) + f^2(v) = f(v) + v \in V_1$

$v - f(v) \in V_{-1} \rightarrow f(v - f(v)) = f(v) - f^2(v) = f(v) - v$

$= -(-f(v) + v) = -(v - f(v)) \in V_{-1}$