Esercitazione 19

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CRITERIO PER STABILIRE SE DEI SSP SONO IN $\oplus$

V $\mathbb{K}$ sp. v. $\quad W_1, \dots, W_K$ ssp di V.

$W_1, \dots, W_K$ sono in somma diretta $\Leftrightarrow$ valgono 1 v 2

$$\left[\begin{array}{l} \underset{\substack{\cap \\ W_1}}{w_1} + \dots + \underset{\substack{\cap \\ W_K}}{w_K} = 0 \Rightarrow w_J = 0 \quad \forall J = 1, \dots, K \end{array}\right.$$

1) $W_1 \cap W_2 = \{0\}, (W_1 + W_2) \cap W_3 = \{0\}, \dots, (W_1 + \dots + W_{K-1}) \cap W_K = \{0\}$

Ogni W_J e' in $\oplus$ con $\sum_{i<J} W_i$ (precedenti)

2) $W_J \cap (W_1 + \dots + \cancel{W_J} + \dots + W_K) = \{0\} \quad \forall J = 1, \dots, K.$

Ogni W_J e' in $\oplus$ con $\sum_{i=1}^{K} W_i \setminus W_J$ (escluso se stesso)

dim

2) $\Rightarrow$

sia $w_J \in W_J \quad w_J = w_1 + \dots + \cancel{w_J} + \dots + w_K \quad w_i \in W_i$

$$\Rightarrow \underset{\substack{\cap \\ W_1}}{w_1} + \dots + \underset{\substack{\cap \\ W_J}}{(-w_J)} + \dots + \underset{\substack{\cap \\ W_K}}{w_K} = 0 \underset{\substack{\Downarrow \\ \text{sono in } \oplus}}{\Rightarrow} w_J = 0$$

per hp.

$\Leftarrow$

Sia $\underset{\substack{\cap \\ W_1}}{w_1} + \dots + \underset{\substack{\cap \\ W_K}}{w_K} = 0$

$$\underset{\substack{\cap \\ W_J}}{w_J} = -(w_1 + \dots + \cancel{w_J} + \dots + w_K) \in W_1 + \dots + \cancel{W_J} + \dots + W_K$$

$\Rightarrow \in \cap W_i = \{0\}$ per hp $\Rightarrow w_J = 0$

1) $\Rightarrow \quad w_J \in W_J, \quad w_J = w_1 + \dots + w_{J-1} + 0 + \dots + 0$

1) $\Rightarrow$ $\underline{w}_J \in W_J$, $\underset{\underset{W_J}{\cap}}{w_J} = \underset{\underset{W_1}{\cap}}{\underline{w_1}} + \dots + \underset{\underset{W_{J-1}}{\cap}}{\underline{w_{J-1}}} + \underset{\underset{W_{J+1}}{\cap}}{\underline{0}} + \dots + \underset{\underset{W_K}{\cap}}{\underline{0}}$

$\Rightarrow \underline{w}_J = 0$ (add zeri e torno al caso 2)

$\Leftarrow$

$\underset{\underset{W_K}{\cap}}{\underline{w_K}} = -(\underline{w_1} + \dots + w_{\underline{K-1}}) \in W_1 + \dots + W_{K-1} \Rightarrow \in \bigcap W_J = \{0\}$

$\Rightarrow \underline{w_K} = 0 \Rightarrow \underline{w_1} + \dots + \underline{w_{K-1}} = \underline{0}$

induzione $\rightsquigarrow$ $\underline{w_{K-1}} = -(\underline{w_1} + \dots + w_{\underline{K-2}}) \Rightarrow w_{\underline{K-2}} = 0$
eccetera.

LEGAME TRA $M^B_B : End(V) \longmapsto M(n, \mathbb{K})$

M^B_B è iso di anelli cioè:

1) $M^B_B(f \circ g) = M^B_B(f)\, M^B_B(g)$

2) $M^B_B(id_V) = I$

3) Se f è invertibile $\Rightarrow M^B_B(f^{-1}) = (M^B_B(f))^{-1}$

Sia $p(t) \in \mathbb{K}[t]$ $\quad p(t) = a_m t^m + \dots + a_1 t + a_0 \quad a_j \in \mathbb{K}$

$p(f) = a_m f^m + \dots + a_1 f^1 + a_0 \underset{\underset{id_V}{\parallel}}{f^0} \in End(V)$

$M^B_B(p(f)) = p(\underbrace{M^B_B(f)}_{A}) = a_m A^m + \dots + a_1 A + a_0 I$.

Esercizio

$f: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ lineare.

$ax^2 + bx + c \longrightarrow (2a - 2b + c)x^2 + (2a - 2b - c)x + a - b + c$

Trovare il pol. caratteristico $p_f(t)$

Trovare 1. pol. caratteristico $p_f(t)$
2. spettro $spec(f)$
3. autospazi relativi agli autovalori

sol

Considero $B = \{x^2, x, 1\}$ base di $\mathbb{R}_2[x]$.

Scrivo $A = m_B^B(f) = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

0

$x^2 \xrightarrow{f} 2x^2 + 2x + 1$ perchè $a=1$, $b=c=0$

$x \xrightarrow{f} -2x^2 - 2x - 1$ perchè $b=1$, $a=c=0$

$1 \longmapsto x^2 - x + 1$ perchè $c=1$, $a=b=0$

1

$p_f(t) = p_A(t) = \det(A - tI) = \det\begin{pmatrix} 2-t & -2 & 1 \\ 2 & -2-t & -1 \\ 1 & -1 & 1-t \end{pmatrix} =$

$C^2 \Rightarrow C^2 + C^1$

$\overset{\uparrow}{=} \det\begin{pmatrix} 2-t & -t & 1 \\ 2 & -t & -1 \\ 1 & 0 & 1-t \end{pmatrix} = -t \det\begin{pmatrix} 2-t & -1 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & 1-t \end{pmatrix} =$

$R_1 \to R_1 - R_2$

$\overset{\uparrow}{=} -t \det\begin{pmatrix} -t & 0 & 2 \\ \cancel{2} & \cancel{-1} & \cancel{-1} \\ 1 & 0 & 1-t \end{pmatrix} = -t \det\begin{pmatrix} -t & 2 \\ 1 & 1-t \end{pmatrix} = -t(-t+t^2-2) =$

$= -t(t-2)(t+1) \Rightarrow p_f(t) = p_A(t) = -t(t-2)(t+1)$

2) $Spec(f) = \{\text{radici di } p_f(t)\} = \{0, -1, 2\}$

nota: 0 è radice, det A è 0 $\Rightarrow$ A non invertibile.

nota 0 è radice, det A è 0 $\Rightarrow$ A noninvertibile.

3) $A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

$\lambda = 0$ $\quad V_0(f)$? $\quad V_0(A) = Ker(A - \underbrace{\lambda}_{0} Id) = Ker(A)$

Cerco rnk (A)

rnk(A) < 3 perché A non è invertibile

trovo un minore con det $\neq 0 \Rightarrow$ rnk A = 2

$\Rightarrow \underbrace{\dim V_0(A)}_{m_g(0)=1} = \dim Ker(A) = 3 - rnk A = 3-2 = 1$.

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in Ker A \Rightarrow Ker A = Span\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$

$V_0(f) = Span(x^2 + x)$ $\quad m_g(0) = 1 = \dim V_0(f)$

$\lambda = -1$ $\quad V_{-1}(A) = Ker(A - (-1 I)) = Ker(A + I)$

$A + I = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ $\rightarrow$ det $\neq 0$

rnk (A+I) < 3 e $\Rightarrow$ rnk (A+I) = 2

$\Rightarrow \dim V_{-1}(A) = m_g(-1) = \dim(Ker(A+I)) = 3-2 = 1$

Considero R2 e R3 lin. indip

$\begin{cases} 2x - y - z = 0 \\ x - y + 2z = 0 \end{cases}$ $\quad \begin{cases} x - 3z = 0 \\ x - y + 2z = 0 \end{cases}$ $\quad \begin{cases} x = 3z \\ 5z = y \end{cases}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z \\ 5z \\ z \end{pmatrix} = z \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \Rightarrow V_{-1}(A) = \mathrm{Span} \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

$$V_{-1}(f) = \mathrm{Span}(3x^2 + 5x + 1) \qquad \dim V_{-1}(f) = 1 = m_g(-1)$$

$\lambda = 2 \qquad V_2(A) = \mathrm{Ker}(A - 2I)$

$$A - 2I = \begin{pmatrix} 0 & -2 & 1 \\ 2 & -4 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$\mathrm{rnk}(A - 2I) < 3 \quad$ e $\quad \mathrm{rnk}(A - 2I) = 2$

$\Rightarrow \dim V_2(A) = m_g(2) = 3 - 2 = 1$

$$\begin{cases} -2y + z = 0 \\ 2x - 4y - z = 0 \\ x - y - z = 0 \end{cases} \qquad \begin{cases} z = 2y \\ 2x - 4y - 2y = 0 \\ x - y - 2y = 0 \end{cases} \qquad \begin{cases} z = 2y \\ +6y = 2x \\ x = 3y \end{cases}$$

$$\begin{cases} z = 2y \\ x = 3y \\ x = 3y \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3y \\ y \\ 2y \end{pmatrix} = y \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$\Rightarrow V_2(A) = \mathrm{Span}(3x^2 + x + 2) \qquad \dim V_2(A) = 1.$

$$V_0(A) \oplus V_{-1}(A) \oplus V_2(A) = \mathbb{R}_2[x]$$

$\subseteq$ vale sempre

qui vale anche $\supseteq$ $\Rightarrow$ f diagonalizzabile.

Esercizio

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ -1 & -2 & 3 \end{pmatrix}$$

$$P_A(t) = \det(A - tI) = \begin{pmatrix} 1-t & 4 & 1 \\ 0 & 1-t & 0 \\ -1 & -2 & 3-t \end{pmatrix} =$$

$$= (1-t)\det\begin{pmatrix} 1-t & 1 \\ -1 & 3-t \end{pmatrix} = (1-t)\big((1-t)(3-t)+1\big) =$$

$$= (1-t)(t^2-4t+4) = (1-t)(t-2)^2$$

$\mathrm{spec}(A) = \{1, 2\}$ $\quad m_a(1) = 1 \quad m_a(2) = 2$

$\qquad\qquad\qquad\qquad\Downarrow \qquad\qquad \Downarrow$

$\qquad\qquad\qquad mg(1) = 1 \quad mg(2) \leq 2.$

Considero $V_2(A) = \mathrm{Ker}(A - 2I) = \begin{pmatrix} \boxed{\begin{matrix} -1 & 4 \\ 0 & -1 \end{matrix}} & \begin{matrix} 1 \\ 0 \end{matrix} \\ \begin{matrix} -1 & -2 \end{matrix} & 1 \end{pmatrix}$

$\mathrm{rnk}(A - 2I) = 2 \Rightarrow mg(2) = 3 - 2 = 1$

$\Rightarrow V_1 \oplus V_2 \subsetneq \mathbb{R}^3$ non è diagonalizzabile.

DIAGONALIZZABILITÀ CON LE MATRICI.

- $A \in M(n, \mathbb{K})$ si dice diagonaliz. se $\exists\, M \in GL(n, \mathbb{K})$ t.c. $MAM^{-1} = D$
- Se $\mathrm{Spec}(A) = \{\lambda\}$ $\quad A$ diagonal. $\Leftrightarrow A = \lambda I$
  $MAM^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$
- Se A ha n autovalori distinti $\Rightarrow A$ è diagonaliz.
  $\hookrightarrow m_a(\lambda_i) = 1 \ \forall i \Rightarrow mg(\lambda_i) = 1 \ \forall i.$

Esercizio

Per quali a, A è diagonalizz?

$$A = \begin{pmatrix} a & 2a+1 & a^2-1 \\ 0 & a^2 & a \\ 0 & 0 & 2a-1 \end{pmatrix} \qquad a \in \mathbb{R}.$$

$$P_A(t) = (a-t)(a^2-t)(2a-1-t)$$

$$\Rightarrow \operatorname{Spec}(A) = \{a, a^2, 2a-1\}$$

$$a = a^2 \iff a = 0, 1$$

$$a = 2a-1 \iff a = 1$$

$$a^2 = 2a-1 \iff a = 1$$

caso 1 $\quad a = 0$

$$\operatorname{Spec}(A) = \{0, -1\}.$$

$m_g \le m_a$

$$m_a(-1) = 1 \Rightarrow m_g(-1) = 1$$

$$m_a(0) = 2 \Rightarrow m_g(0) \le 2$$

$$A_0 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \dim V_0(A) = \ker(A) = 3 - \underset{\operatorname{rnk} A}{\underset{\shortparallel}{2}} = 1$$

$$\Rightarrow m_g(0) = 1$$

In tal caso dato che $\underset{2}{\underset{\shortparallel}{m_a(0)}} \neq \underset{1}{\underset{\shortparallel}{m_g(0)}}$

non è diagonalizzabile.

caso 2 $\quad a = 1$

$$\operatorname{Spec}(A) = \{1\} \qquad A \text{ è diag} \iff A = \lambda I$$

$$A_1 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \neq I$$

$$A - 1 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \neq I$$

$\dim V_1 = 3 - \text{rnk}(A - I) = 1 < 3 = m_a(1)$

No diag.

CASO 3 $a \neq 0, 1$

$\text{Spec}(A) = \{a, a^2, 2a-1\}$ distinti

$\Rightarrow A$ diag. $\quad 1 = m_a(a) = m_a(a^2) = m_a(2a-1)$

$\qquad\qquad\quad 1 = m_g(a) = m_g(a^2) = m_g(2a-1)$

Proposizione

1. $f \in \text{End}(V)$ diag $\Rightarrow f^k$ diag. $\forall k > 0$
2. Se f è invertibile $\Rightarrow f^{-1}$ è diag.

dim

1. f diag $\Rightarrow \exists \; B = \{\underline{v_1}, \dots, \underline{v_n}\}$ base di V di autovettori di f

t.c. $f(\underline{v_j}) = \lambda_j \underline{v_j} \qquad \lambda_j \in \mathbb{K}$.

$f^k(\underline{v_j}) = f^{k-1}(f(\underline{v_j})) = f^{k-1}(\lambda_j \underline{v_j}) = \lambda_j f^{k-1}(\underline{v_j}) =$

$= \text{———} = \lambda_j^k \underline{v_j}$

- B è anche base di autovettori di f^k
- $\text{Spec}(f^k) = \{\lambda_1^k, \dots, \lambda_n^k\}$

2) Se f è invertibile $\Rightarrow \lambda_j \neq 0 \quad \forall j$.

$\underline{v_j} = f^{-1}(\lambda_j \underline{v_j}) \Rightarrow f^{-1}(\underline{v_j}) = \dfrac{\underline{v_j}}{\lambda_j} = \dfrac{1}{\lambda_j} \underline{v_j}$

$$\underline{v}_j = f^{-1}(\lambda_j \underline{v}_j) \Rightarrow f^{-1}(\underline{v}_j) = \frac{\underline{v}_j}{\lambda_j} = \frac{1}{\lambda_j}\underline{v}_j$$

- B è anche base di autovettori per f^{-1}
- $\operatorname{Spec}(f^{-1}) = \left\{ \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n} \right\}$