Esercitazione 22

martedì 11 gennaio 2022 23:38

POL. MINIMO RELATIVO

$\dim P(f,v) = \deg \mu_{f,v} = d$ e $v, f(v), \dots, f^{d-1}(v)$ sono una base B detta base ciclica di $P(f,v)$

$P(f,v)$ è f-invariante e $m_B^B(f|_{P(f,v)}) = \begin{pmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & & & \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & a_{d-1} \end{pmatrix}$

dove $f^d(v) = a_0 v + \dots + a_{d-1} f^{d-1}(v) \Rightarrow \mu_{f,v} = t^d - (a_{d-1}t^{d-1} + \dots + a_0) = \pm p_{f|_{P(f,v)}}$

esempio metodo 3

$\mathbb{K} = \mathbb{R}$ $\quad A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{pmatrix}$ Trovare μ_A

Uso la base canonica di $\mathbb{R}^3 = \{e_1, e_2, e_3\}$.

$e_1 \xmapsto{A} 2e_1 + e_2 + e_3 \xmapsto{A} 2(2e_1 + e_2 + e_3) + -e_2 + 3e_3 + 3e_2 - e_3$

$= 4e_1 + 4e_2 + 4e_3 = 4(\underbrace{2e_1 + e_2 + e_3}_{A \cdot e_1}) - 4e_1$

comb lin di

$A^2 e_1 = 4Ae_1 - 4e_1 \Rightarrow (A^2 - 4A + 4I)e_1 = 0$

$\mu_{A,e_1} = t^2 - 4t + 4 = (t-2)^2$

$e_2 \xmapsto{A} -e_2 + 3e_3 \xmapsto{A} -(-e_2 + 3e_3) + 3(3e_2 - e_3) =$

$= 10e_2 - 6e_3 = -2(-e_2 + 3e_3) + 8e_2$

$A^2 e_2 = -2Ae_2 + 8e_2 \qquad \mu_{A,e_2} = t^2 + 2t - 8 = (t+4)(t-2)$

$e_3 \xmapsto{A} 3e_2 - e_3 \xmapsto{A} 3(-e_2 + 3e_3) - (3e_2 - e_3) =$

$= -3e_2 + 9e_3 - 3e_2 + e_3 = -6e_2 + 10e_3 = -2(3e_2 - e_3) + 8e_3$

$\Rightarrow A^2 e_3 = -2Ae_3 + 8e_3 \qquad \mu_{A, e_3} = t^2 + 2t - 8 = (t+4)(t-2)$

$\mu_A = mcm\left((t-2)^2, (t+4)(t-2)\right) = (t+4)(t-2)^2$

$\mu_A \mid p_A$ ma $\deg p_A = 3 \Rightarrow \mu_A = \pm p_A$.

Proposizione

$\exists \underline{v} \in V$ t.c. $v, f(v), f^2(v), \dots, f^k(v)$ lin. indip $\Rightarrow$

$\deg \mu_f \geq k+1$

se $\deg \mu_f = \dim V = \deg p_f \Rightarrow \mu_f = \pm p_f$.

es di prima

$\deg \mu_A = 3 = \deg p_A = \dim \mathbb{R}^3 \Rightarrow \mu_A = -p_A = -(t+4)(t-2)^2$.

Proposizione

$\exists \underline{v} \in V$ t.c. $v, f(v), \dots, f^{n-1}(v)$ base di $V \Rightarrow \mu_f = \pm p_f$

Vale anche $\Leftarrow$

dim.

Supp $\exists \underline{v} \in V$ t.c. $\mu_{f,v} = \mu_f$

- Se $\boxed{\mathbb{K}\ \infty}$ considero $S = \{\mu_{f,v}$ t.c. $\underline{v} \in V\}$ è un insieme finito perchè $\mu_{f,v} \mid \mu_f$ e $S \subset \{$divisori di $\mu_f\}$ $\hookrightarrow$ finito.

$\exists \underline{v_1}, \dots, \underline{v_k} \in V$ t.c. $S = \{\mu_{f,v_1}, \dots, \mu_{f,v_k}\}$

$\underline{v} \in V \quad \mu_{f,v} \in S \quad \exists\ 1 \leq j \leq k$ t.c. $\mu_{f,v} = \mu_{f,v_j}$

$0 = \mu_{f,v}(f)(v) \quad \Rightarrow v \in \operatorname{Ker} \mu_{f,v_j}(f)$

ma $V = \bigcup^k \operatorname{Ker}(\mu_{f,v_i}(f)) \Rightarrow \exists\, i$ t.c. $V = \operatorname{Ker} \mu_{f,v_i}(f)$

ma $V = \bigcup_{i=1}^{k} \mathrm{Ker}(\mu_{f,v_j}(f)) \Rightarrow \exists i$ t.c. $V = \mathrm{Ker}\,\mu_{f,v_i}(f)$

$\Rightarrow \mu_{f,v_i} \in I(f) \to \mu_f \mid \mu_{f,v_i} \Rightarrow \mu_f = \mu_{f,v_i}$.

- $\boxed{\mathbb{K} \text{ finito}}$ servono dei lemmi

Lemma

Sia $\underline{v} \in V$ t.c. $\mu_{f,v} = p_1 p_2$ $\;p_1, p_2 \in \mathbb{K}[t]$ moniche e $(p_1, p_2) = 1$

sia $w_1 = p_1(f)(\underline{v})$. Allora $\mu_{f,\underline{w_1}} = p_2$

dim

$$p_2(f)(w_1) = p_2(f)(p_1(f)(\underline{v})) = (p_2(f) \circ p_1(f))(\underline{v}) =$$

$$= (p_2 \cdot p_1)(f)(\underline{v}) = \mu_{f,v}(f)(v) = 0 \Rightarrow p_2 \in I(f, \underline{w_1}) \Rightarrow$$

$$\mu_{f,w_1} \mid p_2$$

Viceversa

Per Bezout $\exists h_1, h_2 \in \mathbb{K}[t]$ t.c. $1 = h_1 p_1 + h_2 p_2$

applico $f \to \mathrm{id}_V = h_1(f) p_1(f) + h_2(f) p_2(f) \Rightarrow$

applico $v \longrightarrow v = h_1(f) \underbrace{p_1(f)(v)}_{w_1} + h_2(f) p_2(f)(v) =$

$$= h_1(f)(w_1) + (h_2 p_2)(f)(\underline{v}) \Rightarrow$$

$$(\mu_{f,w_1} p_1)(f)(\underline{v}) = 0$$

$$(\mu_{f,w_1} p_1 h_1)(f)(w_1) = (p_1 h_1)(f)(\mu_{f,w_1}(f)(w_1))$$

"0

$(\mu_{f,w_1} p_1 h_1)(f)(w_1) = (p_1 h_1)(f)(\mu_{f,w_1}(f)(w_1))$ $= 0$

$$(\mu_{f,w_1} \underbrace{p_1 h_2 p_2}_{\mu_{f,v}})(f)(\underline{v}) = (\mu_{f,w_1} h_2)(f)(\mu_{f,\underline{v}}(f)(\underline{v})) = 0.$$

$\Rightarrow \mu_{f,w_1} p_1 \in I(f,v) \Rightarrow \mu_{f,v} \mid \mu_{f,w_1} p_1$, $\mu_{f,v} = p_1 p_2$

$\Rightarrow p_2 \mid \mu_{f,w_1} \Rightarrow \mu_{f,w_1} = p_2.$

Lemma

$\forall v, w \in V \quad \exists z \in V$ t.c. $\mu_{f,z} = \text{mcm}(\mu_{f,v}, \mu_{f,w})$

$\mu_{f,v} = \underbrace{q_1^{m_1} \cdots q_k^{m_k}}_{p_1} \underbrace{q_{k+1}^{m_{k+1}} \cdots q_h^{m_h}}_{p_2}$ $\quad q_i \in K[t]$ irrid. distinti monici

$\mu_{f,w} = \underbrace{q_1^{n_1} \cdots q_k^{n_k}}_{a_2} \underbrace{q_{k+1}^{n_{k+1}} \cdots q_h^{n_h}}_{a_1}$

$m_i \geq n_i \qquad m_i < n_i$

$\text{mcm}(\mu_{f,v}, \mu_{f,w}) = p_1 a_1$

$(p_1, a_1) = 1 = (p_2, a_2) = (p_1, p_2) = (a_1, a_2)$

$z = p_2(f)(\underline{v}) + a_2(f)(w)$

uso lemma $p_1 = \mu_{f, p_2(f)(v)}$

$$(p_1 a_1)(f)(z) = (p_1 a_1 p_2)(f)(v) + (\underbrace{p_1 a_1 a_2})(f)(w) = 0$$

$$(p_1 a_i)(f)(z) = (\underbrace{p_1 a_1 p_2}_{\mu_{f,v}})(f)(v) + (\underbrace{p_1 a_1 a_2}_{\mu_{f,w}})(f)(w) = 0$$

$$\overset{\shortparallel}{0} \qquad \overset{\shortparallel}{0}$$

$$\Rightarrow p_1 a_1 \in I(f, z) \Rightarrow \mu_{f,z} \mid p_1 a_1$$

$$p_2(f)(v) = z - a_2(f)(w)$$

$$(\mu_{f,z}\, a_1)(f)(p_2(f)(v)) = (\mu_{f,z}\, a_i)(f)(z) - \mu_{f,z}(\overset{\mu_{f,w}}{a_1 a_2})(f)(w)$$

$$\overset{\shortparallel}{0} \qquad \overset{\shortparallel}{0} = 0$$

$$\Rightarrow \mu_{f,z}\, a_1 \in I(f, p_2(f)(v)) \Rightarrow p_1 = \mu_{f, p_2(f)(v)} \mid \mu_{f,z}\, a_1$$

$(a_1, p_1) = 1$

$$\Rightarrow p_1 \mid \mu_{f,z}$$

Analogamente $a_1 \mid \mu_{f,z} \Rightarrow p_1 a_1 \mid \mu_{f,z} \Rightarrow \mu_{f,z} = p_1 a_1$

Adesso prendo $v_1, \dots, v_m$ gener. di V.

$$\mu_1 = \text{mcm}(\mu_{f,v_1}, \mu_{f,v_2}, \dots, \mu_{f,v_m}) =$$

$$= \text{mcm}(\underbrace{\text{mcm}(\mu_{f,v_1}, \mu_{f,v_2})}_{\mu_{f,z_1}}, \dots, \mu_{f,v_m}) =$$

$$= \text{mcm}(\mu_{f,z_1}, \mu_{f,v_3}, \dots, \mu_{f,v_m}) = \mu_{f,z_m}.$$

Proposizione

Se W è f-invariante $f^k|_W = (f|_W)^k \Rightarrow f|_W \in \text{End}(W)$

$\forall p \in \mathbb{K}[t] \quad p(f|_W) = p(f)|_W.$

Allora $I(f|_W) = \{p \in \mathbb{K}[t] \text{ t.c. } p(f|_W) = 0\} =$

Allora $I(f|_W) = \{p \in \mathbb{K}[t]$ t.c. $p(f|_W) = 0\} =$

$= \{p \in \mathbb{K}[t]$ t.c. $p(f)(W) = \{\underline{0}\}\}$

$= p \in \mathbb{K}[t]$ t.c. $\ker p(f) \supset W \} \supset I(f)$

$\Rightarrow \mu_{f|_W} \mid \mu_f.$