Esercitazione 7

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APPLICAZIONI LINEARI DA $\mathbb{K}^n$ a $\mathbb{K}^m$

$$L: M(m, n, \mathbb{K}) \longmapsto Hom(\mathbb{K}^n, \mathbb{K}^m)$$
$$A \longmapsto L_A$$

dove $L_A : \mathbb{K}^n \longmapsto \mathbb{K}^m$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 A^1 + \dots + x_n A^n$$

$$Imm(L_A) = Span(A^1, \dots, A^n)$$

$$Ker(L_A) = \{ \underline{x} \in \mathbb{K}^n \mid A\underline{x} = \underline{0} \}$$

Esercizio 1

$g: \mathbb{R}^3 \longmapsto \mathbb{R}^2$ $\qquad g = L_A$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y-z \\ 2x+2y-2z \end{pmatrix}$$

· Trovare A

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y-z \\ 2x+2y-2z \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{A^1} + y\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{A^2} + z\underbrace{\begin{pmatrix} -1 \\ -2 \end{pmatrix}}_{A^3}$$

$$A = (A^1, A^2, A^3) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \end{pmatrix}$$

· Trovare $Im(g) = Span(A^1, A^2, A^3) = Span\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

· Trovare $Ker(g) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x+y-z=0 \\ \underbrace{2x+2y-2z=0}_{\text{superflua.}} \end{matrix} \right\}$

esercizio 2

$f : \mathbb{R}^2 \mapsto \mathbb{R}^3$ t.c. $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ e $\underline{v_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$f(v_1) = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$ $f(v_2) = \begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix}$

- $v_1 + v_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = 4\,E^2 \Rightarrow E^2 = \frac{1}{4}(v_1 + v_2)$
- $v_1 - v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\,E^1$ $E^1 = \frac{1}{2}(v_1 - v_2)$

- $f(e_1) = \frac{1}{2} f(v_1 - v_2) = \frac{1}{2} f(v_1) - \frac{1}{2} f(v_2) = \frac{1}{2}\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

- $f(e_2) = \frac{1}{4} f(v_1 + v_2) = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

$f = L_A$ $A = \begin{pmatrix} 1 & -3 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$ $f : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x - 3y \\ 2x + y \\ x + y \end{pmatrix}$

$\mathrm{Im} f = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}\right)$ non è surg è piano in $\mathbb{R}^3$

$\mathrm{Ker} f = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x - 3y = 0 \\ 2x + y = 0 \\ x + y = 0 \end{array} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \Rightarrow f$ inj.

Esercizio 3

Trovare $\mathrm{Im} f \cap \mathrm{Ker}\, g$

Quaderno.

Proposizione

$g \circ f = 0 \iff \mathrm{Im} f \subset \mathrm{Ker}\, g$ $f : V \to W$ $g : W \to Z$.

$\Rightarrow$

$\underline{w} \in \mathrm{Im} f$ tesi $\to \underline{w} \in \mathrm{Ker}\, g$

$\underline{w} \in \mathrm{Im} f \Rightarrow \exists\ \underline{v} \in V$ t.c. $f(\underline{v}) = \underline{w}$; hp

$\underline{w} \in \mathrm{Im} f \Rightarrow \exists\ \underline{v} \in V$ t.c. $f(\underline{v}) = \underline{w}$

applico $g \rightarrow g(\underline{w}) = g(f(\underline{v})) = (g \circ f)(\underline{v}) \overset{\text{hp}}{=} 0 \Rightarrow$

$g(\underline{w}) = 0$ e cioè $\underline{w} \in \mathrm{Ker}\, g$

$\Leftarrow$ $\underline{v} \in V$. $(g \circ f)(\underline{v}) = g(f(\underline{v})) = 0 \Rightarrow g \circ f = 0$

$f(\underline{v}) \in \mathrm{Im} f \subseteq \mathrm{Ker}\, g \Rightarrow g(f(\underline{v})) = 0.$