Applicazioni lineari (08/11/2022)

$f: V \to W$ (V, W spazi su $\mathbb{K}$) è **app. lineare** se:

(i) $\forall \underline{v_1}, \underline{v_2} \in V$, $f(\underline{v_1} + \underline{v_2}) = f(\underline{v_1}) + f(\underline{v_2})$

(ii) $\forall \underline{v} \in V, \alpha \in \mathbb{K}$, $f(\alpha \underline{v}) = \alpha f(\underline{v})$

- $f(\underline{0}) = f(\underline{0}) + f(\underline{0}) \Rightarrow f(\underline{0}) = \underline{0}$
- $\mathrm{Im}\, f$ è sottospazio di W.
- $\ker f$ è sottospazio di V.
- f surgettiva $\Leftrightarrow \mathrm{Imm}\, f = W$
- f iniettiva $\Leftrightarrow \ker f = \{\underline{0}\}$
  - f iniettiva $\Rightarrow \ker f = \{\underline{0}\}$ (banale)
  - $\ker f = \{\underline{0}\}$, $f(\underline{v}) = f(\underline{w}) \Rightarrow f(\underline{v} - \underline{w}) = \underline{0} \Rightarrow$
    $\Rightarrow \underline{v} = \underline{w}$. $\boxtimes$
- $\underbrace{f: \mathbb{K} \to \mathbb{K} \text{ è lineare}}_{(1)} \Leftrightarrow \underbrace{\exists a \in \mathbb{K} \mid f(x) = ax \ \forall x \in \mathbb{K}}_{(2)}$
  - $(1) \Leftarrow (2)$: banale
  - $(1) \Rightarrow (2)$: $a = f(1_{\mathbb{K}})$. $f(x \cdot 1_{\mathbb{K}}) = x f(1_{\mathbb{K}}) = ax$. $\boxtimes$
  - $a = 0 \Rightarrow \ker f = \mathbb{K}$, $\mathrm{Im}\, f = \{0\}$
    $a \neq 0 \Rightarrow \ker f = \{0\}$, $\mathrm{Im}\, f = \mathbb{K}$

es. $f: \mathbb{R}^2 \longrightarrow M(2,\mathbb{K})$, $\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} a-b & a+2b \\ 2a-b & 2a+2b \end{pmatrix} =$

$= a\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + b\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$ è app. lineare

$f\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{cases} a-b=0 \Rightarrow a=b \\ a+2b=0 \Rightarrow 3b=0 \Rightarrow b=0 \\ 2a-b=0 \\ 2a+2b=0 \end{cases}$ $\quad \operatorname{Ker} f = \{\underline{0}\}$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \overset{?}{\in} \operatorname{Im} f \begin{cases} a-b=1 \Rightarrow a=1+b \Rightarrow \frac{2}{3} \\ a+2b=0 \Rightarrow 3b=-1 \Rightarrow b=-\frac{1}{3} \\ 2a-b = \frac{4}{3}+\frac{1}{3} = \frac{5}{3} \neq 0 \ \lightning \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \operatorname{Im} f$

$\dim \mathbb{R}^2 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

$\Downarrow$

$\operatorname{Im} f = \operatorname{Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\right)$ $\quad \dim \operatorname{Im} f = 2$

$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \operatorname{Im} f \Longleftrightarrow \begin{cases} a-b=x \\ a+2b=y \\ 2a-b=z \\ 2a+2b=t \end{cases}$ è risolubile $\Longleftrightarrow$

$\Longleftrightarrow \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ è risolubile $\Longleftrightarrow$

$\Longleftrightarrow \operatorname{rg} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}}_{A} = \operatorname{rg} \underbrace{\begin{bmatrix} 1 & -1 & x \\ 1 & 2 & y \\ 2 & -1 & z \\ 2 & 2 & t \end{bmatrix}}_{\tilde{A}}$ (teorema di Rouché-Capelli)

$U = \operatorname{Span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad f(U) = \lambda f\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} -1 & 5 \\ 0 & 6 \end{pmatrix} =$

$= \operatorname{Span}\begin{pmatrix} -1 & 5 \\ 0 & 6 \end{pmatrix}$

$$U = \mathrm{Span}(\overbrace{\underline{w_1}, \underline{w_2}, \dots, \underline{w_n}}^{W}) \qquad W \text{ genera } U \Rightarrow$$

$$f(U) = f(\alpha_1 \underline{w_1} + \dots + \alpha_n \underline{w_n}) = \qquad \Rightarrow f(W) \text{ genera}$$

$$= \alpha_1 f(\underline{w_1}) + \dots + \alpha_n f(\underline{w_n}) \text{ "="} \qquad f(U)$$

$$\text{"="} \mathrm{Span}(f(\underline{w_1}), \dots, f(\underline{w_n})) = \mathrm{Span}(f(W))$$

Quindi se G genera V spazio, $f(G)$ genera $f(V) =$ $= \mathrm{Im}\, f$. Pertanto nel caso di prima ($\mathbb{R}^2$) bastava prendere $f\begin{pmatrix}1\\0\end{pmatrix}$ e $f\begin{pmatrix}0\\1\end{pmatrix}$ per ottenere dei generatori.

es. $V = \mathbb{K}[x]$, $W = \mathbb{K}$ $\quad a \in \mathbb{K}$

$V_a : \mathbb{K}[x] \to \mathbb{K}$, $p \mapsto p(a)$ é lineare

- $V_a(p_1 + p_2) = (p_1 + p_2)(a) = p_1(a) + p_2(a) =$
  $= V_a(p_1) + V_a(p_2)$. ✓
- $V_a(\alpha p) = (\alpha p)(a) = \alpha p(a) = \alpha V_a(p)$. ✓

- $\mathrm{Im}\, V_a = \mathbb{K}$ $\quad (V_a(k) = k)$
- $\ker V_a = \{ p \in \mathbb{K}[x] \mid p = (x-a)\, q(x) \}$ (ideale)
  - $(x-a), (x-a)x, \dots, (x-a)x^n, \dots$ base

es. $T : M(m,n,\mathbb{K}) \longrightarrow M(n,m,\mathbb{K})$

$A \longmapsto A^T$ (matrice trasposta)

e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Ossia $A = (a_{ij})_{\substack{i=1\to m \\ j=1\to n}} \iff A^T = (a_{ji})_{\substack{i=1-m \\ j=1-n}}$

- $(A+B)^T = \left((a_{ij})_{\substack{i=1\to m \\ j=1\to n}} + (b_{ij})_{\substack{i=1\to m \\ j=1\to n}} \right)^T =$

$= \left((a_{ij}+b_{ij})_{\substack{i=1\to m \\ j=1\to n}} \right)^T =$

$= (a_{ji}+b_{ji})_{\substack{i=1\to m \\ j=1\to n}} = A^T + B^T$

- $(\alpha A)^T = (\alpha a_{ji})_{\substack{i=1\to m \\ j=1-n}} = \alpha A^T$

Quindi è app. lineare.
Inoltre $(A^T)^T = A \implies A^T$ è surgettiva.

$A^T = \underline{0} \implies A = \underline{0} \implies \ker A^T = \{\underline{0}\} \implies A^T$ è iniettiva.

Quindi A^T è biunivoca, ossia è un isomorfismo naturale.

es.

$$tr: M(m,n,\mathbb{K}) \longrightarrow \mathbb{K}$$

$$A = (a_{ij})_{\substack{i=1\to m \\ j=1\to n}} \longmapsto \sum_{i=1}^{\min(k,n)} a_{ii} \quad \text{(traccia)}$$

- $tr(A+B) = tr(A) + tr(B)$
- $tr(\alpha A) = \alpha\, tr(A)$

- $\operatorname{Im} tr = \mathbb{K}$
- $\ker tr$
  - $E_{ij} \in \ker tr$ con $i \neq j$, e ve ne sono $mn - k$ $(k = \min(m,n))$
  - $E_{11} - E_{ii}$ $_{i=2-k}$ $\in \ker tr$ (sono $k-1$)
  - Poiché $\dim \ker tr < mn$ (in quanto sottospazio proprio), osservando che le scorse matrici sono lin. ind., si ottiene $mn - 1 \leq \dim \ker tr < mn$, quindi $\dim \ker tr = mn - 1$.
  - Allora tali matrici formano una base e $\dim \ker tr = mn - 1$. Quest'ultimo risultato era in realtà prevedibile:

$$\dim \ker f = \dim M(m,n,\mathbb{K}) - \dim \overbrace{\operatorname{Im} f}^{\mathbb{K}} =$$

$$= mn - \underbrace{\dim \mathbb{K}}_{1} = mn - 1.$$