es. A antisimmetrica $\in M_n(\mathbb{K})$ (i.e. $A^T = -A$) $\Rightarrow$

$\Rightarrow$ $\mathrm{rg}(A) \equiv 0 \ (2)$ con $\mathrm{char}\,\mathbb{K} \neq 2$

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \\ -a_{12} & 0 & & \\ -a_{13} & & \ddots & \\ & & & \ddots \end{pmatrix}$$

$\forall A$, $\mathrm{rg}(A) = \mathrm{rg}(A^T) \Rightarrow$

$\Rightarrow$ si può fare la riduzione di Gauss per colonna (è equivalente a farla per riga sulla trasposta)

$$(MAM^T)^T = M^{T^T} A^T M^T = M(-A)M^T = -MAM^T,$$

quindi MAM^T è antisimmetrica.

Per induzione su n.

base: $A = (0)$, $\mathrm{rg}\, A = 0$

passo induttivo:

se $A = \begin{pmatrix} 0 & 0 & \cdots \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{A'}} \\ \vdots & & \end{pmatrix}$, anche A' è antisimmetrico

$$\mathrm{rg}(A) = \mathrm{rg}(A') \Rightarrow \mathrm{rg}(A) \equiv 0 \ (2)$$

altrimenti:

$$A = \begin{array}{c} \\ \\ i \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & & & j \end{array} \\ \begin{pmatrix} 0 & & & a \\ & \ddots & & \\ -a & & & \ddots \end{pmatrix} \end{array} \xrightarrow[C_j \leftrightarrow C_2]{R_i \leftrightarrow R_2} \begin{pmatrix} 0 & a & & \\ -a & 0 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \xrightarrow[C_2 \to \frac{1}{a} C_2]{R_2 \to \frac{1}{a} R_2} \quad (a \neq 0)$$

$$1 \longrightarrow \begin{array}{cc} & \begin{array}{cccc} & & & j \end{array} \\ \begin{array}{c} \\ \\ \\ i \end{array} & \begin{pmatrix} 0 & 1 & \cdots & b \\ -1 & 0 & & c \\ \vdots & & \ddots & \\ -b & -c & & \ddots \end{pmatrix} \end{array} \xrightarrow[\substack{R_i \to R_i + cR_1 \\ C_i \to C_i + cC_1}]{\substack{R_i \to R_i - bR_2 \\ C_j \to C_j - bC_2}} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \multicolumn{2}{c}{\underbrace{\boxed{\ddots}}_{\text{antisimmetrica}}} \\ 0 & 0 & & \end{pmatrix} \longrightarrow \cdots$$

$$\xrightarrow{\cdots} \begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & \cdots & \cdots \\ 0 & \vdots & \multicolumn{2}{c}{\underbrace{\boxed{\ddots}}_{\text{antisimmetrica}}} \\ \vdots & \vdots & & \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 1 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \multicolumn{4}{c}{\underbrace{\boxed{A'}}_{\text{ancora antisimmetrica}}} \\ 0 & 0 & & & & \\ \vdots & \vdots & & & & \end{pmatrix}$$

$$\operatorname{rg}(A) = \operatorname{rg}(A') + 2 \Rightarrow \operatorname{rg}(A) \equiv 0 \ (2) \qquad \boxtimes$$

Prop. $f : V \to W$ lineare manda basi di V in $W \Rightarrow f$ isomorfismo

$\dim V = n$

$\underline{v_1}, \dots, \underline{v_n}$ base di $V \Rightarrow f(\underline{v_1}), \dots, f(\underline{v_n})$ base di $W \Rightarrow$

$\Rightarrow \dim W = n$

$\operatorname{Span}(f(\underline{v_1}), \dots, f(\underline{v_n})) = \operatorname{Imm} f$, $\dim \operatorname{Imm} f = n \Rightarrow$

$\Rightarrow \operatorname{Imm} f = W$ ✓

$\dim \ker f = \dim V - \dim \operatorname{Imm} f = 0$ ✓

$\boxtimes$

es. $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 10 \end{pmatrix} \in M_2(\mathbb{R})$

Se generano, anche le loro coordinate generano $\mathbb{R}^4$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 3 & -1 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & -1 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & -1 & 10 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rg(A) = 3 \Rightarrow$

$\Rightarrow$ non generano

es. $\begin{pmatrix} 2 \\ -1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 \\ 5 \\ 3 \\ -1 \end{pmatrix}}_{\text{lin. ind. con gli altri}}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -3 & 2 \\ -1 & 1 & 1 \\ -3 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & 5 \\ -1 & 1 & 0 \\ -3 & 1 & 0 \end{pmatrix} \to$$

$$\to \underbrace{\begin{pmatrix} 2 & -1 & 5 \\ -1 & 0 & 0 \\ -3 & -2 & 0 \end{pmatrix}}_{A} \qquad rg(A) = 3$$

Quindi i vettori sono tutti lin. ind. tra loro e generano $\mathbb{R}^4$.

App. lineari

$h(\operatorname{Ker} f) \subset \operatorname{Ker} g$ $\qquad$ $\underline{x} \in \operatorname{Ker} f$, $g(h(\underline{x})) = l(f(\underline{x})) = \underline{0}$ ✓

$l(\operatorname{Im} f) \subset \operatorname{Im} g$ $\qquad$ $l(f(\underline{x})) = g(h(\underline{x}))$ ✓

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "h"'] & W \arrow[d, "l"] \\ V' \arrow[r, "g"'] & W' \end{tikzcd}$$

$g \circ h = l \circ f$

(diagramma commutativo)

Se sono isomorfismi, vale l'uguaglianza considerando gli inversi.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "{[\underline{v}]_B}"'] & W \arrow[d, "{[\underline{v}]_{B'}}"] \\ \mathbb{K}^n \arrow[r] & \mathbb{K}^m \end{tikzcd}$$

$[\underline{v}]_{B'} \circ f \circ [\underline{v}]_B^{-1} = f_A$ $\qquad$ $A = M^B_{B'}(f) = \left[[f(\underline{v}_1)]_{B'} | \dots \right]$

es. $f : M_2(\mathbb{R}) \longrightarrow \mathbb{R}_4[t]$

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b)t^4 + (a-b+c-d)t^3 + (a+b+2c-2d)t^2 + $$
$$+ (a-b+3c+3d)t + (a+b+4c-4d)1$$

$$M_D^B(f) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 2 & -2 \\ 1 & -1 & 3 & 3 \\ 1 & 1 & 4 & -4 \end{pmatrix} \longrightarrow$$

$$\longrightarrow A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & -1 \\ 1 & 0 & 3 & 4 \\ 1 & 2 & 4 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & 4 & -3 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & 4 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & -3 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad rg(A) = 4 \implies$$

$$\implies \dim \operatorname{Ker} f = \dim \operatorname{Ker} f_A = 0 \implies$$

$$\implies f \text{ iniettiva}$$