es. A antisimmetrica $\in M_n(\mathbb{K})$ (i.e. $A^T = -A$) $\Rightarrow$

$\Rightarrow \operatorname{rg}(A) \equiv 0 \ (2)$ con $\operatorname{char} \mathbb{K} \neq 2$

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \\ -a_{12} & 0 & & \\ -a_{13} & & \ddots & \\ & & & \ddots \end{pmatrix}$$

$\Rightarrow$ $\forall A$, $\operatorname{rg}(A) = \operatorname{rg}(A^T) \Rightarrow$ si può fare la riduzione di Gauss per colonna (è equivalente a farla per riga sulla trasposta)

$$(MAM^T)^T = M^{T^T} A^T M^T = M(-A)M^T = -MAM^T,$$

quindi MAM^T è antisimmetrica.

Per induzione su n.

base: $A = (0)$, $\operatorname{rg} A = 0$

passo induttivo:

se $A = \begin{pmatrix} 0 & 0 & \cdots \\ 0 & & \\ \vdots & & A' \end{pmatrix}$, anche A' è antisimmetrico

$\operatorname{rg}(A) = \operatorname{rg}(A') \Rightarrow \operatorname{rg}(A) \equiv 0 \ (2)$

altrimenti:

$$A = \begin{array}{c} \\ \\ \\ i \end{array} \begin{array}{c} \begin{array}{cccc} & & j & \end{array} \\ \begin{pmatrix} 0 & & \boxed{a} & \\ & \ddots & & \\ & & \ddots & \\ \boxed{-a} & & & \ddots \end{pmatrix} \end{array} \xrightarrow[C_j \leftrightarrow C_2]{R_i \leftrightarrow R_2} \begin{pmatrix} 0 & a & \\ -a & 0 & \\ & & \ddots \\ & & & \ddots \end{pmatrix} \xrightarrow[C_2 \to \frac{1}{a} C_2]{R_2 \to \frac{1}{a} R_2} \quad (a \neq 0)$$

$$1 \longrightarrow \begin{array}{c|c} & \begin{matrix} & & j \end{matrix} \\ i & \begin{pmatrix} 0 & 1 & \cdots & b \\ -1 & 0 & & c \\ \vdots & & \ddots & \\ -b & -c & & \ddots \end{pmatrix} \end{array} \xrightarrow[\substack{R_i \to R_i + cR_1 \\ C_i \to C_i + cC_1}]{\substack{R_i \to R_i - bR_2 \\ C_j \to C_j - bC_2}} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \multicolumn{2}{c}{\underbrace{\ddots}_{\text{antisimmetrica}}} \\ 0 & 0 & & \end{pmatrix} \longrightarrow \cdots$$

$$\cdots \longrightarrow \begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & \cdots & \\ 0 & \vdots & \multicolumn{2}{c}{\underbrace{\ddots}_{\text{antisimmetrica}}} \\ \vdots & \vdots & & \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 1 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & & & & \\ 0 & 0 & \multicolumn{4}{c}{\underbrace{A'}_{\text{ancora antisimmetrica}}} \\ 0 & 0 & & & & \end{pmatrix}$$

$$rg(A) = rg(A') + 2 \Rightarrow rg(A) \equiv 0 \ (2) \qquad \square$$

Prop. $f: V \to W$ lineare manda basi di V in $W \Rightarrow f$ isomorfismo

$\dim V = n$

$\underline{v_1}, \dots, \underline{v_n}$ base di $V \Longrightarrow f(\underline{v_1}), \dots, f(\underline{v_n})$ base di $W \Longrightarrow$

$\Longrightarrow \dim W = n$

$\text{Span}(f(\underline{v_1}), \dots, f(\underline{v_n})) = \text{Imm} f$, $\dim \text{Imm} f = n \Longrightarrow$

$\Longrightarrow \text{Imm} f = W$ ✓

$\dim \ker f = \dim V - \dim \text{Imm} f = 0$ ✓

$\square$

es. $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 10 \end{pmatrix} \in M_2(\mathbb{R})$

Se generano, anche le loro coordinate generano $\mathbb{R}^4$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 3 & -1 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & -1 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & -1 & 10 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rg(A) = 3 \Rightarrow$

$\Rightarrow$ non generano

es. $\begin{pmatrix} 2 \\ -1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 \\ 5 \\ 3 \\ -1 \end{pmatrix}}_{\text{lin. ind. con gli altri}}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -3 & 2 \\ -1 & 1 & 1 \\ -3 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & 5 \\ -1 & 1 & 0 \\ -3 & 1 & 0 \end{pmatrix} \to$$

$$\to \underbrace{\begin{pmatrix} 2 & -1 & 5 \\ -1 & 0 & 0 \\ -3 & -2 & 0 \end{pmatrix}}_{A} \qquad rg(A) = 3$$

Quindi i vettori sono tutti lin. ind. tra loro e generano $\mathbb{R}^4$.

App. lineari

$h(\ker f) \subset \ker g$ $\qquad$ $\underline{x} \in \ker f,\ g(h(\underline{x})) = l(f(\underline{x})) = \underline{0}$ ✓

$l(\operatorname{Im} f) \subset \operatorname{Im} g$ $\qquad$ $l(f(\underline{x})) = g(h(\underline{x}))$ ✓

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "h"'] & W \arrow[d, "l"] \\ V' \arrow[r, "g"'] & W' \end{tikzcd}$$

$g \circ h = l \circ f$

(diagramma commutativo)

Se sono isomorfismi, vale l'uguaglianza considerando gli inversi.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "{[\underline{v}]_B}"'] & W \arrow[d, "{[\underline{v}]_{B'}}"] \\ \mathbb{K}^n \arrow[r] & \mathbb{K}^m \end{tikzcd}$$

$[\underline{v}]_{B'} \circ f \circ [\underline{v}]_B^{-1} = f_A$ $\qquad$ $A = M^B_{B'}(f) = \left[[f(\underline{v_1})]_{B'} | \dots \right]$

es. $f: M_2(\mathbb{R}) \longrightarrow \mathbb{R}_4[t]$

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b)t^4 + (a-b+c-d)t^3 + (a+b+2c-2d)t^2 + (a-b+3c+3d)t + (a+b+4c-4d)1$$

$$M_D^B(f) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 2 & -2 \\ 1 & -1 & 3 & 3 \\ 1 & 1 & 4 & -4 \end{pmatrix} \longrightarrow$$

$$\longrightarrow A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & -1 \\ 1 & 0 & 3 & 4 \\ 1 & 2 & 4 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & 4 & -3 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & 4 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & -3 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad rg(A) = 4 \implies$$

$\implies \dim \ker f = \dim \ker f_A = 0 \implies$

$\implies f$ iniettiva

es. V, W, Z sp. vett. su $\mathbb{K}$. $f \in \mathcal{L}(V,W)$ $g \in \mathcal{L}(V,Z)$

$\exists\, h \in \mathcal{L}(V,Z) \mid \; g \circ h = f$

- $\ker g \subset \ker f \qquad f(\underline{v}) = h(g(\underline{v})) = h(\underline{0}) = \underline{0}$
- $\dim V = n$, dim

Sia $\underline{v_1}, \dots, \underline{v_k}$ base di $\ker g$ e $\underline{v_1}, \dots, \underline{v_k}, \dots, \underline{v_n}$ il suo completamento su V. $g(\underline{v_{k+1}}), \dots, g(\underline{v_n})$ è base di $\text{Imm}\, g$

Sia $\underline{z_i} = g(\underline{v_i})$.

$$h \begin{cases} \underline{z_{k+1}} \mapsto f(\underline{v_{k+1}}) \\ \dots \\ \underline{z_n} \mapsto f(\underline{v_n}) \end{cases}$$

es. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ lineare, $f(U) \subset W$, $f(W) \subset U$

$U = \{ 2x - y + z = 0 \}$ $\quad W = \{ y + z = 0 \}$

$\dim U = \dim W = 2$ $\quad \dim (U \cap W) = 1$

$\dim (U + W) = 3$

$$U \cap W \begin{cases} 2x - y + z = 0 \Rightarrow x = -z \\ y + z = 0 \Rightarrow y = -z \end{cases} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$U \cap W = \text{Span} \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{\underline{v_1}}$$

$\underline{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ lin. ind. con $\underline{v_1}$ base di U

$\underline{v_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ lin. ind. con $\underline{v_1}$ base di W

$\{ \underline{v_1}, \underline{v_2}, \underline{v_3} \}$ base di $\mathbb{R}^3$

$f(\underline{v_1}) \in U \cap W \Rightarrow f(\underline{v_1}) = a\, \underline{v_1}$

$f(\underline{v_2}) \in W \implies f(\underline{v_2}) = b\underline{v_1} + c\underline{v_3}$

$f(\underline{v_3}) \in U \implies f(\underline{v_3}) = d\underline{v_1} + e\underline{v_2}$

$$M_B^B(f) = \begin{pmatrix} a & b & d \\ 0 & 0 & e \\ 0 & c & 0 \end{pmatrix}$$

$\cdot\ f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $\qquad \underline{e_3} = \frac{1}{2}(\underline{v_1} + \underline{v_2} + \underline{v_3})$

$$[f(e_3)]_B = \begin{pmatrix} a & b & d \\ 0 & 0 & e \\ 0 & c & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}_B = \frac{1}{2}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}_B = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a & b & d \\ 0 & 0 & e \\ 0 & c & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies$$

$$\rightarrow \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} a+b+d \\ e \\ c \end{pmatrix} \longrightarrow \begin{cases} c = 4 \\ e = 1 \\ a+b+d = 0 \end{cases}$$

$\cdot\ \dim V_2(f) = 2 \implies \dim \operatorname{Ker}(f - 2\,Id) = 2$

$$M_B^B(f - 2\,Id) = M_B^B(f) - 2\,M_B^B(Id) =$$

$$= \begin{pmatrix} a & b & d \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} a-2 & b & d \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{pmatrix} = A$$

$$\dim \operatorname{Ker} A = 2 \implies rg(A) = 1$$

OSS. $V_\lambda(f) = \{ \underline{v} \mid f(\underline{v}) = \lambda \underline{v} \} = \operatorname{Ker}(f - \lambda Id_{\mathbb{R}^3})$

es. V, W sp. vett. su $\mathbb{K}$. $\dim V = n$, $\dim W = m$. $H \subset V$, $K \subset W$ ssp.

$T = \{ f \in \mathcal{L}(V,W) \mid \operatorname{Ker} f \supset H \wedge \operatorname{Im} f \subset K \}$ è un sottospazio di $\mathcal{L}(V,W)$.

$$B \to \underbrace{\underbrace{\underline{v_1}, \dots, \underline{v_h}}_{\text{base di } H}, \dots, \underline{v_n}}_{\text{base di } V} \qquad D \to \underbrace{\underbrace{\underline{w_1}, \dots, \underline{w_k}}_{\text{base di } K}, \dots, \underline{w_m}}_{\text{base di } W}$$

$$M_D^B(T) = \left(\begin{array}{cccc|cc} 0 & 0 & & 0 & \multicolumn{2}{c}{C} \\ \cline{5-6} 0 & 0 & \dots & 0 & & \\ \vdots & \vdots & & \vdots & 0 & 0 \\ & & & & \vdots & \vdots \end{array} \right)$$

$\underbrace{\qquad\qquad}_{h \text{ volte}}$ (i vettori di H sono nel ker di f)

$m-k$ volte (Im $f \subset K$, quindi nessun vettore si scrive usando $\underline{w_{k+1}}, \dots, \underline{w_m}$)

$$C \in M(k, n-h, \mathbb{K}) \implies \dim T = k(n-h)$$

OSS. $g \circ f$ lineare.

$$\mathrm{Imm}\, g|_{\mathrm{Imm}\, f} = \mathrm{Imm}\,(g \circ f)$$

$$\mathrm{Ker}\, g|_{\mathrm{Imm}\, f} = \mathrm{Ker}\, g \cap \mathrm{Imm}\, f$$

$$\underbrace{\dim \mathrm{Imm}\, f}_{rg(f)} = \underbrace{\dim \mathrm{Imm}\,(g \circ f)}_{rg(g \circ f)} + \dim(\mathrm{Ker}\, g \cap \mathrm{Imm}\, f)$$

$$rg(g \circ f) = rg(f) - \dim(\mathrm{Ker}\, g \cap \mathrm{Imm}\, f)$$

$$\dim \mathrm{Ker}(g \circ f) = \dim \mathrm{Ker}\, f + \dim(\mathrm{Ker}\, g \cap \mathrm{Imm}\, f)$$

- g iniettiva:
  - $rg(g \circ f) = rg(f)$
  - $\dim \mathrm{Ker}\,(g \circ f) = \dim \mathrm{Ker}\, f$

- f surgettiva:
  - $rg(g \circ f) = \dim \mathrm{Dom}(g) - \dim \mathrm{Ker}\, g = rg(g)$
  - $\dim \mathrm{Ker}(g \circ f) = \dim \mathrm{Ker}\, f + \dim \mathrm{Ker}\, g$

es.

$f : \mathbb{R}^3 \to \mathbb{R}^2$
$\underline{x} \mapsto A\underline{x}$

$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix} = M^{e_{\mathbb{R}^2}}_{e_{\mathbb{R}3}}(f)$

$B = (\underline{e_1} - \underline{e_2}, \underline{e_2} + \underline{e_3}, \underline{e_3})$

$M^{B}_{e_{\mathbb{R}3}}(f) = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 5 & 2 \end{pmatrix}$

$D = \left(\begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$

$M^{e_{\mathbb{R}^2}}_{D}(f) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

$M^{B}_{D}(f) = M^{e_{\mathbb{R}3}}_{D}(Id_{\mathbb{R}3})\, M^{e_{\mathbb{R}2}}_{e_{\mathbb{R}3}}(f)\, M^{B}_{e_{\mathbb{R}_2}}(Id_{\mathbb{R}2})$