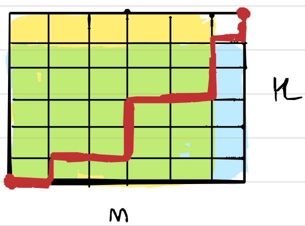
# More examples

## (i) Lattice paths




In a $m \times k$ grid there are $\binom{m+k}{k}$ lattice paths (if you identify the letter H with an horizontal move and V with a vertical one, you're counting words in $\Sigma = \{H, V\}$ with length $m+k$ and $m$ H and $k$ V).

## ii)

$$\sum_{m\geq 0} \overbrace{\sum_{k\geq 0} \binom{m}{k} x^k}^{(1+x)^m} y^m = \frac{1}{1-(1+x)y} = \frac{1}{1-y} \; \frac{1}{1-\frac{y}{1-y}x} =$$

$$= \sum_{k\geq 0} \frac{y^k}{(1-y)^{k+1}} x^k .$$

$$\sum_{k\geq 0} \frac{y^k}{(1-y)^{k+1}} x^k = \sum_{k\geq 0} \left( \sum_{m\geq 0} \binom{m}{k} y^m \right) x^k \quad \xrightarrow{\text{equivalence coeff.-wise}}$$

$$\rightsquigarrow \frac{y^k}{(1-y)^{k+1}} = \sum_{m\geq 0} \binom{m}{k} y^m .$$

By shifting one gets:

$$\frac{1}{\underbrace{(1-y)^k}} = \sum_{m\geq 0} \binom{m+k-1}{m} y^m .$$

("stars and bars")

$(1+y+\cdots)(1+y+\cdots) \Rightarrow$ this counts $k$-multisubsets

## iii)

$$\frac{1}{1-x_1 t} \; \frac{1}{1-x_2 t} \cdots \frac{1}{1-x_k t} =$$

$$= (1+x_1 t + \cdots)(1+x_2 + \cdots) \cdots =$$

$$= \sum_{m\geq 0} t^m \left( \sum_{a_1+\cdots+a_k = m} x_1^{a_1} \cdots x_k^{a_k} \right)$$

This is the (complete) homogeneous symmetric polynomial $h_m(x_1, \ldots, x_k)$.

There are $\binom{m+k-1}{m}$ monomials (check $\frac{1}{(1-x)^k}$).

iii) $A_m \triangleq \{$ words in $\Sigma = \{1,2\}$ That sum to $m\}$

For example, $112 \in A_4$.

Let's find a recursion:

- $$|A_m| = \underbrace{|A_{m-1}|}_{\text{if it ends in } 1} + \underbrace{|A_{m-2}|}_{\text{" in } 2} \qquad (1)$$

- $|A_0| = 0$, $|A_1| = 1$. Hence we get The Fibonacci numbers! Let's derive Them. Set $F_m \triangleq |A_m|$.

Let $\underbrace{F(x) = \sum_{m \geq 0} F_m x^m}_{\text{gen. fun.}}$. From (1), we get:

$$\underbrace{\sum_{m \geq 2} F_m x^m}_{F(x) - F_0 - F_1 x} = \underbrace{\sum_{m \geq 2} F_{m-1} x^m}_{x(F(x) - F_0)} + \underbrace{\sum_{m \geq 2} F_{m-2} x^m}_{x^2 F(x)} \implies$$

$$\implies F(x) - x = x F(x) + x^2 F(x) \implies \boxed{F(x) = \frac{x}{1 - x - x^2}}.$$

By using partial fractions:

$$F(x) = \frac{1}{r^+ - r^-}\left(\frac{1}{1 - r^+ x} - \frac{1}{1 - r^- x}\right),$$

where $r^+ = \frac{1+\sqrt{5}}{2} = \varphi$, $r^- = \frac{1-\sqrt{5}}{2}$.

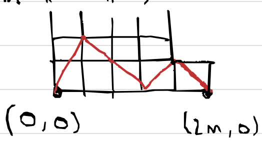
hence, by equalling coefficients:

$$\boxed{F_m = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right)}.$$

$$\sim \frac{\varphi^m}{\sqrt{5}}$$

iv) $C_m = \#\{\text{Dyck paths of length } 2m\}$

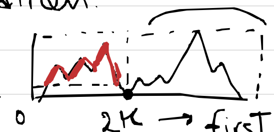
(in $\mathbb{N}\times\mathbb{N}$)



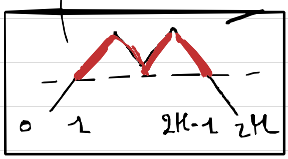

Rules for Dyck paths
- you can go diagonally $(+(1,1)$ or $+(1,-1))$
- you can't go below the $x$-axis
- must start from $(0,0)$, must end in $(2m,0)$

Let's look for a recurrence relation:

$$\begin{cases} C_m = \displaystyle\sum_{k=1}^{m} C_{k-1}\cdot C_{m-k} & (2) \\ C_0 = 1 \end{cases}$$




$2k \rightarrow$ first time it touches the ground after 0




There is only one way to mix up the left path and the right one ... use the multiplicative principle of counting and you're good to go!

Let $c(x) = \sum_{i\geq 0} c_i x^i$. From (2) we get:

$$\sum_{m\geq 1} C_m x^m = \sum_{m\geq 1}\sum_{k=1}^{m} C_{k-1} C_{m-k} x^m =$$

$$= x \underbrace{\sum_{m\geq 1}\sum_{k=1}^{m} C_{k-1}x^{k-1} \cdot C_{m-k}x^{m-k}}_{C(x)^2} \Longrightarrow$$

$$\Longrightarrow C(x) - 1 = x\, C(x)^2 \xrightarrow{\text{we take the neg. sol.}} \boxed{C(x) = \frac{1-\sqrt{1-4x}}{2x}} =$$

$$= \frac{1}{2x}\left(1 - (1-4x)^{1/2}\right) =$$

$$= -\frac{1}{2x}\sum_{m\geq 1}\binom{1/2}{m}(-4)^m x^m, \qquad \text{(mind the shift!)}$$

hence $$C_m = -\frac{1}{2}\binom{1/2}{m+1}(-4)^{m+1} =$$

$$= -\frac{1}{2}\binom{1/2}{m+1}(-1)^{m+1}\, 2^{2(m+1)} = (-1)^m\binom{1/2}{m+1}\, 2^m\, 2^{m+1} =$$

$$= \frac{(-1)^m}{m+1}\, 2^m \cdot 2^{m+1}\, \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-m\right)}{m!} =$$

$$= \frac{1}{n+1} \frac{(2n)!}{n!\,n!} = \boxed{\frac{1}{n+1} \binom{2n}{n}}$$ Catalan's numbers

This Term can be explained with pure combinatorics by linking lattice and Dyck paths Together.

## Formal power series

- $\sum_m a_m x^m + \sum_m b_m x^m \triangleq \sum_m (a_m + b_m) x^m$ (ADDITION)

- $(\sum_m a_m x^m)(\sum_m b_m x^m) \triangleq \sum_m \left(\sum_{k=0}^{m} a_k b_{m-k}\right) x^m$ (MULTIPLICATION)

Let's characterize units, i.e. $f$ s.t. $\exists g$ s.t. $fg = 1$
We say $g$ is the inverse/reciprocal of $f$.

$\rightarrow$ $x$ is not a unit, since $x \cdot f$ can't have $1$ as a constant

Prop. $f(x) = \sum_{m \geq 0} a_m x^m \in A[[x]]$ has a reciprocal iff $a_0 \in A^*$.

($\Rightarrow$) Trivial by computation.

($\Leftarrow$) Suppose $a_0 \in A^*$. We look for $g(x) = \sum_m b_m x^m$ s.t. $fg = 1$, i.e.

$$1 = \left(\sum_m a_m x^m\right)\left(\sum_m b_m x^m\right) = \underbrace{a_0 b_0}_{\substack{\text{hence we} \\ \text{set } b_0 = a_0^{-1}}} + (b_1 a_0 + a_1 b_0) x +$$

$$+ (b_2 a_0 + b_1 a_1 + b_0 a_2) x^2 + \dots \Longrightarrow$$

$$\Longrightarrow \begin{cases} a_0 b_0 = 1 \longrightarrow \text{unique solution } \checkmark \\ b_1 a_0 + a_1 b_0 = 0 \Longrightarrow b_1 = -a_0^{-1} a_1 b_0 \\ b_2 a_0 + b_1 a_1 + b_0 a_2 \Longrightarrow b_2 = -a_0^{-1}(b_1 a_1 + b_0 a_2) \\ \dots \end{cases}$$

$\square$

We would like to implement composition $(f(g(x)))$...
It should be $f(g(x)) = \sum_{m \geq 0} a_m (g(x))^m$, but there might be series as terms, which cannot be allowed

- if $f(x)$ is a polynomial, composition makes sense
- if $g(x)$ has no constant term, it makes sense as well (there are finite terms for each $x^m$)

We also look for "functional inverses" (or compositional), i.e. $g$ st.

$$f(g(x)) = x = g(f(x)).$$

For that to happen, $f$ must not have constant terms for sure. If such $g$ exists, then $g(0)=0$ as well.

**Prop** $f(x) = \sum_{n\geq 0} a_n x^n$ with $f(0)=0$. Then a comp inv. exists iff $a_1 \in A^*$, and it is unique*

**Notation**: We write $f^{\langle -1\rangle}(x)$ for the comp. inverse.

$\rightsquigarrow$ $f(g(x)) = x$ is enough to check the invertibility (compos.)**

$$f(g(x)) = a_1(b_1 x + b_2 x^2 + \dots) + a_2(b_1 x + \dots)^2 + \dots =$$

$$= \underbrace{a_1 b_1}\, x + \underbrace{(a_1 b_2 + a_2 b_1^2)}\, x^2 + \dots$$

$\Longrightarrow$ $b_1$ must be $a_1^{-1}$, $=0$, so I get $b_2$ ... (as before)

** Since $b_1 \in A^*$, $g$ has a right inverse, which is unique; let's call it $\tilde{f}$

$$g(\hat{f}(x)) = x, \quad \text{and} \quad f(g(\hat{f}(x)) = \hat{f}(x))$$

$$f(x) \quad ✓$$

□

$\rightarrow$ comp. units form a group!

* hence $f$ has a comp. inv. iff $f(0)=0$ and $a_1 \in A^*$.