Es. sui sistemi lineari

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- $\exists!\, W \subseteq V$ ssp. di dimensione minima (i.e. $W = \{\underline{0}\}$, $\dim W = 0$).
- $\exists!\, W \subseteq V$ ssp. di dim. massima (i.e. $W = V$, $\dim W = \dim V$).

$$\Updownarrow$$

$$W \subseteq V \wedge \dim W = \dim V \iff$$
$$\iff W = V$$
$$\left(B_W = B_V \Rightarrow w \in V,\ (W \subseteq V) \atop v \in Span(B_W) \to (V \subseteq W) \right)$$

Sistemi lineari

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = \\ \quad \cdots \\ \quad \cdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = \end{cases} \boxed{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}$$

Le soluzioni del sistema sono un sottospazio di $\mathbb{K}^n$. Supponiamo che x_i sia ricavabile in funzione di $x_{k+1}, \dots, x_n$.

$$\begin{cases} x_1 = f_1(x_{k+1}, \dots, x_n) \\ \quad \dots \end{cases}$$

es.

$$H = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix} \in \mathbb{R}^5 \;\middle|\; \begin{matrix} x + y - z - 2t = 0 \\ y - z + 2t + w = 0 \end{matrix} \right\}$$

$$\begin{cases} x = 2t + z - y \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = t \overbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}^{w_1} + z \overbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}}^{w_2} +$$

$$\begin{cases} x = 2t + z - y \\ w = z - 2t - y \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix} = t\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + y\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}}_{\underline{w_3}}$$

$H = \mathrm{Span}(\underline{w_1}, \underline{w_2}, \underline{w_3})$,

dove $\underline{w_1}, \underline{w_2}$ e $\underline{w_3}$ sono base (essendo linearmente indipendenti).

es. 2 $\quad \underline{v_1} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \\ -4 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -3 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{v_4} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -2 \end{pmatrix} \quad \underline{v_5} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad \underline{v_6} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$K = \mathrm{Span}(\underline{v_1}, \dots, \underline{v_6})$

Si cerca un insieme lin. ind.

- $\underline{v_1}$ è lin. ind.
- $\underline{v_1}, \underline{v_2}$ sono lin. ind.
- $\underline{v_1}, \underline{v_2}, \underline{v_3}$ NON sono lin. ind.

$\underline{v_3} = \underline{v_2} - \underline{v_1}$ (si rimuove $\underline{v_3}$)

- $\underline{v_1}, \underline{v_2}, \underline{v_4}$ sono lin. ind.
- $\underline{v_1}, \underline{v_2}, \underline{v_4}, \underline{v_5}$

$$\begin{cases} 2a + 3b = 1 \Rightarrow 4 - 3 = 1 \checkmark \\ a + c = 0 \Rightarrow a = -c \Rightarrow c = -a = -2 \\ -a - b - c = 1 \Rightarrow -\cancel{a} - b + \cancel{a} = 1 \Rightarrow b = -1 \\ a + b = 1 \Rightarrow a = 1 - b = 2 \\ -4a - 3b - 2c = -1 \Rightarrow -8 + 3 + 4 = -1 \checkmark \end{cases} \Rightarrow$$

$\rightarrow \underline{v_5} = 2\underline{v_1} - \underline{v_2} - 2\underline{v_4}$ **lin. dip.** (si rimuove $\underline{v_5}$)

- $\underline{v_1}, \underline{v_2}, \underline{v_4}, \underline{v_6}$

lin. dip. (si rimuove $\underline{v_6}$)

$$\begin{cases} 2a + 3b = -2 \Rightarrow 4 - 6 = -2 \ \checkmark \\ a + c = 1 \Rightarrow a = 1 - c = 2 \\ -a - b - c = 1 \Rightarrow -c = 1 \Rightarrow c = -1 \\ a + b = 0 \Rightarrow a = -b \Rightarrow b = -a = -2 \\ -4a - 3b - 2c = 0 \Rightarrow -8 + 6 + 2 = 0 \ \checkmark \end{cases}$$

Quindi: $\dim(\mathrm{Span}(\underline{v_2}, \dots, \underline{v_6})) = 3$.

$K = \mathrm{Span}(\underline{v_1}, \underline{v_2}, \underline{v_4})$

- $K \subseteq H$, $H = \mathrm{Span}(\underline{w_1}, \underline{w_2}, \underline{w_3})$
  ossia $\underline{v_1} \in H$, $\underline{v_2} \in H$, $\underline{v_3} \in H$.
  Poiché $\dim K = \dim H = 3$, $H = K$.

  In particolare, è semplice riscrivere $\underline{v_i}$ come comb. lin. di $\underline{w_1}, \underline{w_2}, \underline{w_3}$, poiché contengono delle basi canoniche.

Estendiamo K a una base:

- poiché $\dim \mathbb{R}^5 = 5$, ci servono $5 - 3 = 2$ vettori lin. ind. con H.
- $\underbrace{\underline{v_4} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{(\underline{v_1}, \dots, \underline{v_4}) \text{ lin. ind.}}$, per esempio (basta che non appartenga ad H...)
- $\underline{v_5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, per esempio

Alternativamente

- si usa l'algoritmo di estrazione su

$$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\} \cup \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots \right\}}_{\text{BASE CANONICA}}$$

$$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\} \text{ o } \underbrace{\left\{\begin{pmatrix} \vdots \\ 0 \end{pmatrix}, \dots\right\}}_{\text{BASE CANONICA}}$$

· e.g. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ è lin. ind., etc...

es. 3

$$\underbrace{T(3,2,\mathbb{K})}_{\text{triang. sup.}} \ni \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}$$

una base è:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots$$

quindi: $\dim(T(3,2,\mathbb{K}))$

$T(m,n,\mathbb{K})$

$$m \geq n \longrightarrow \underbrace{\overbrace{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}}^{n}}_{\text{e.g.}} \quad \dim = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$n \geq m \longrightarrow \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \dim.$$