Linear Algebra notes

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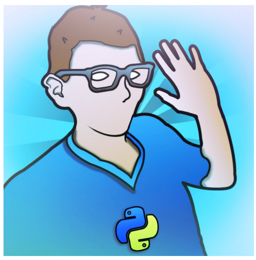

Contents

Chapter 1

Basic Notions.

1.1 Vector spaces

Definition A vector space V is an object made of two sets: the set of the vectors, with its elements being denoted by a lowercase bold letter (e.g. $\mathbf{u}$), and the set of the scalars ($\boldsymbol{K}$), usually denoted by a Greek letter. $\mathbf{u}$ and $\mathbf{v}$ usually denote general vectors, while α and β general scalars.

There are eight properties which vectors hold:

1. Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \ \ \forall\, \mathbf{u}, \mathbf{v} \in V$;
2. Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \ \ \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. Zero vector: there exists a unique vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v} \ \ \forall\, \mathbf{v} \in V$;
4. Inverse vector: there always exists a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{v} - \mathbf{v} = \mathbf{0} \ \ \forall\, \mathbf{v} \in V$;
5. Neutral scalar for multiplication: there always exists a scalar 1 such that $1\mathbf{v} = \mathbf{v} \ \ \forall\, \mathbf{v} \in V$;
6. Multiplicative associativity: $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v}) \ \ \forall\, \mathbf{v} \in V$ and $\forall \alpha, \beta$ which are scalars;
7. Distributive property: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} \ \ \forall\, \mathbf{u}, \mathbf{v} \in V$ and $\forall \alpha$ which is scalar;
8. Vectorial distributive property: $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v} \ \ \forall\, \mathbf{v} \in V$ and $\forall \alpha, \beta$ which are scalars.

   If a vector space is made of scalars which are real numbers, it is called a *real* vector space. Likewise, for complex numbers, it is called a *complex* vector space.

Matrices & polynomials A space $M_{m\times n}$ (sometimes written as $M_{m,n}$) denotes a vector space made of $m \times n$ matrices. A space $\mathbb{P}_n$ denotes a vector space made of n degree polynomials.

A matrix A is an array of m rows and n columns and it is written as $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \ddots & \ddots & a_{2n} \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix}$.

A^T represents the transposed matrix (i.e. a matrix A for which the rows & columns have been inverted). If $A^T = A$, then the matrix is *symmetric*. If $A^T = -A$, it's *antisymmetric*.

1.2 Linear combinations

Definition Given a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \dots, \mathbf{v}_n$ and a collection of scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots, \alpha_n$, the sum $\sum\limits_{p=1}^{n} \alpha_p \mathbf{v}_p$ is called a *linear combination*.

If there exists a unique representation for the *linear combination* of a set of vectors such that $\forall\, \mathbf{v} \in V,\ \mathbf{v} = \sum_{p=1}^{n} \alpha_p \mathbf{v}_p$, these ones are called a *basis*, while their coefficients are called *coordinates*.

A set of vectors is a *basis* if $\mathbf{v} = \sum_{p=1}^{n} x_p \mathbf{v}_p$ admits a unique set of solutions.

Standard basis $\mathbb{F}^n$ represents a general vector space whose vectors has got n coordinates.

The set of vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n$, with $\mathbf{e}_n$ being a vector whose coordinates are all 0 except the n-th, which is equal to 1, is the *standard basis* in $\mathbb{F}^n$.

Likewise, the set of polynomial vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n$, with $\mathbf{e}_n := x^n$, is the *standard basis* in $\mathbb{P}_n$.

1.3 Systems

Definition A system of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_n$ is called a *generating* (or *complete*, or even *spanning*) system if $\forall\, \mathbf{v} \in V, \exists\, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n \in \boldsymbol{K} : \mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$.

Any basis is a generating system.

Linearly independent system A system of vectors is linearly independent when $\mathbf{0}$ is represented by a linear combination where $a_k = 0\ \forall k$, which is called *trivial*. Otherwise, it is called a linearly dependent system.

There always exists a vector in a linearly dependent system which can be represented as a non-trivial linear combination of the other vectors.

A generating system of vectors which is linearly dependent is always a *basis*. Any generating system contains a basis.