Counting problems

I set of indices (usually $I = \mathbb{N}$)

$\forall i \in I$, A_i finite set. We define $a_i \triangleq |A_i|$.

Q: Compute $\{a_i\}_{i \in I}$.

Example $I = \mathbb{N}$

i) $A_n = \{ B \subseteq [n] \}$ where $[n] \triangleq \{1, 2, \dots, n\}$

It is a practical answer. Easily computable.

$a_n = 2^n$ ✓

There are many ways to get this answer. A combinatorial way is to show that $|A_n| = |\{ \text{words of length } n \text{ in } \Sigma = \{0,1\}\}|$.

ii) $D_n = \{ \sigma \in S_n \mid \sigma \text{ has no fixed point} \}$

$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$ ✓?

It has some computational issue. Is it that better than $\sum_{\sigma \in D_n} 1$? "Not a great answer".

iii) $n \in \mathbb{N}$, $n \geq 1$

$\lambda \vdash n \overset{\Delta}{\Longleftrightarrow} \lambda = (\lambda_1, \dots, \lambda_m) \mid \lambda_i \in \mathbb{N} \ \forall i$, and $\lambda_1 \geq \dots \geq \lambda_m > 0$, $\sum_i \lambda_i = n$.

partition

$A_n \triangleq \{ \lambda \vdash n \}$

$a_n = ?$ $\rightsquigarrow$ very complicated answer.

$\rightsquigarrow$ sometimes we don't have good answers for a_n, but we have some explicit relationships for them.

$\{a_m\}_{m=0}^{\infty}$, $a_m \in \mathbb{R}$

$\{a_m\}_{m=0}^{\infty}$ is associated to $a_0 + a_1 x + a_2 x^2 + \dots = \sum_{i=0}^{\infty} a_i x^i$, a formal power series. That is called an ordinary generating function (o.g.f)

Example

- $a_m = 0 \quad \forall m > N \rightsquigarrow$ polynomial
- $a_m = 1 \quad \forall m \rightsquigarrow 1 + x + x^2 + \dots = \frac{1}{1-x}$ (in $\mathbb{R}[[x]]$ — algebraically)
- $a_m = \frac{1}{m!} \rightsquigarrow \sum_{m \geq 0} \frac{x^m}{m!} \overset{\Delta}{=} e^x$

$\rightsquigarrow$ composition of o.g.f.'s makes sense only in some special cases. For example, e^{1+x} makes no sense; let's try to compute $[x]\, e^{1+x}$.

$$e^{1+x} = \sum_{m \geq 0} \frac{(1+x)^m}{m!} = 1 + (1+x) + \frac{(1+2x+x^2)}{2} + \frac{(1+3x+\dots)}{6} + \dots$$

$$\Longrightarrow [x]\, e^{1+x} \text{ "="} \sum_{i \in \mathbb{N}} i \text{, non-sense.}$$

Binomial coefficient

$$\binom{m}{k} \overset{\Delta}{=} \#\{B \subseteq [m] \mid |B| = k\}$$

$$\binom{m}{0} = 1 \qquad \binom{m}{1} = m \qquad \binom{m}{2} = \frac{m(m-1)}{2} \quad \dots$$

$\binom{m}{k} = \frac{m!}{k!(m-k)!}$; let's try to derive this formula from its o.g.f.:

$$a_k^{(m)} \overset{\Delta}{=} \binom{m}{k} \longrightarrow f_m(x) = \sum_{k \geq 0} \binom{m}{k} x^k = \sum_{k=0}^{m} \binom{m}{k} x^k \quad \left[\binom{m}{k} = 0 \text{ for } k > m\right]$$

Claim: $\boxed{f_m(x) = (1+x)^m}$ (Newton's binomial Theorem)

1. $\binom{m}{k} = \underbrace{\binom{m-1}{k-1}}_{m \in} + \underbrace{\binom{m-1}{k}}_{m \notin} \longrightarrow$ recurrence relation

2. $\sum_{k\geq 1} \binom{m}{k} x^k = \sum_{k\geq 1} \binom{m-1}{k-1} x^k + \sum_{k\geq 1} \binom{m-1}{k} x^k$

$\rightsquigarrow \quad f_m(x) - 1 = f_{m-1}(x)\, x + f_{m-1}(x) - 1 \Longrightarrow$

$\Longrightarrow \quad f_m(x) = (1+x)\, f_{m-1}(x) = \dots = (1+x)^m.$ ✓

To get the formula for $\binom{m}{k}$, we derive both expressions:

- $\binom{m}{k} = \underbrace{[x^k]\, f(x)}_{k\text{-th coeff.}} = \left. \dfrac{\left(\frac{d}{dx}\right)^k f(x)}{k!} \right|_{x=0} = \dfrac{m!}{(m-k)!\,k!}$ ✓

$\rightsquigarrow$ Through this formula we can def. $\binom{\cdot}{k}$ on $\mathbb{C}$.

$$\binom{\alpha}{k} \triangleq \frac{(\alpha)_k}{k!} \quad \text{where} \quad (k$$

$$(\alpha)_k \triangleq \begin{cases} \alpha(\alpha-1)\cdots(\alpha-k+1) & \text{if } k\geq 1 \\ 1 & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases} \quad (k\in\mathbb{Z})$$

This lets us def. $(1+x)^\alpha$ on $\alpha\in\mathbb{C}$ as well with the binomial coeff.

Exercise

$$(a+b)_m = \sum_{k=0}^{m} \binom{m}{k} (a)_k\, (b)_{m-k} \in \mathbb{R}[[a,b]].$$