Calculus of ordinary generating functions (o.g.f)

$f(x) = \sum_i a_i x^n \implies f'(x) \stackrel{\Delta}{=} \sum_{m\geq 1} m a_i x^{n-1}$ (The formal derivative)

- $(f(x)\, g(x))' = f'(x)\, g(x) + f(x)\, g'(x)$ (product rule)
- $(f(g(x)))' = f'(g(x))\, g'(x)$ (chain rule)

[integration makes sense]

- if $f' = 0$, Then f is a constant
- $f' = f \implies f = a_0 e^x$.

We will use D for the differential operator.

We're associating sequences to their o.g.f's.

$\{a_m\}_{m\in\mathbb{N}} \xleftrightarrow{\text{o.g.f.}} f(x) = \sum_{m\geq 0} a_m x^m$, $\{b_m\}_{m\in\mathbb{N}} \xleftrightarrow{\text{o.g.f.}} g$.

- $\{a_{m+n}\}_{m\in\mathbb{N}} \longleftrightarrow \left(f - \sum_{m=0}^{n-1} a_m x^m\right) / x^n$ (shifting)
- $\{a_m + b_m\}_{m\in\mathbb{N}} \longleftrightarrow f + g$ (sum)
- $\{m a_m\}_{m\in\mathbb{N}} \longleftrightarrow x \cdot D f$

In general $\boxed{\{p(m) \cdot a_m\}_{m\in\mathbb{N}} \longleftrightarrow \overbrace{p(x\cdot D)}^{\text{operator}} f}$

Example Let $f(x) = 1 + x + \dots + x^m = \dfrac{x^{m+1} - 1}{x - 1}$

Then $g(x) = \sum_{j=0}^{m} j^k x^j = (xD)^k f(x)$

$$\implies \sum_{j=0}^{m} j^k = (xD)^k \left.\frac{x^{m+1} - 1}{x - 1}\right|_{x=1}.$$

- $\left\{\sum_{k=0}^{m} a_k b_{m-k}\right\}_{m\in\mathbb{N}} \longleftrightarrow f g$ (product)
- $\left\{\sum_{m_1+\dots=m} a_{m_1} \cdots a_{m_k}\right\}_{m\in\mathbb{N}} \longleftrightarrow f^k$ (power)

Example $$\frac{1}{(1-x)^k} = \sum_{m\geq 0}\Big(\sum_{m_1+\dots=m} 1\Big)x^m = \sum_{m\geq 0}\binom{m+k-1}{m}x^m.$$

- $$\boxed{\frac{1}{1-x}f \longleftrightarrow \Big\{\sum_{k=0}^{m} a_k\Big\}_{m\geq 0}}$$ (partial sums)

Example $$\sum_{k=0}^{m} F_k = F_{m+2}-1$$

$$\frac{1}{1-x}F(x) = \frac{x}{(1-x)(1-x-x^2)} \checkmark$$

$$\frac{F(x)-x}{x^2} - \frac{1}{1-x}$$

$$= \frac{1}{x^2}\Big(\frac{x}{1-x-x^2} - x\Big) - \frac{1}{1-x}$$

$$= \frac{x}{(1-x)(1-x-x^2)} \checkmark$$

Example $$\sum_{m,k}\binom{m}{k}x^k y^m = \frac{1}{1-(x+1)y} \Longrightarrow$$

$$\Longrightarrow x\sum_{m,k}\binom{m}{k}x^k x^m = \frac{x}{1-x-x^2} = \sum_{m\geq 0}F_m x^m \Longrightarrow$$

$$\Longrightarrow \boxed{F_r = \sum_{m+k=r-1}\binom{m}{k}}$$

Let $A(n,k) = \#\{\text{set partitions of } [n] \text{ into } k \text{ non-empty parts}\}$

Example $n=3$

- $A(3,1) = 1 \quad \{\{\{1,2,3\}\}\}$
- $A(3,2) = \{\{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{2,3\},\{1\}\}\}$
- $A(3,3) = \{\{\{1\},\{2\},\{3\}\}\}$

We use $\left\{ {n \atop k} \right\} \triangleq |A(n,k)|$ (Stirling numbers of the second kind)

- $\left\{ {n \atop k} \right\} = 0$ for $k > n$,
- $\left\{ {n \atop 0} \right\} = 0$ for $n \geq 1$,
- $\left\{ {0 \atop 0} \right\} = 1$

Let's look for a recurrence relation:

$$\left\{ {n \atop k} \right\} = \underbrace{\left\{ {n-1 \atop k-1} \right\}}_{\substack{n \text{ is in} \\ \text{a singleton.}}} + \underbrace{k \left\{ {n-1 \atop k} \right\}}_{\substack{\text{for each partition in } A(n-1,k), \\ \text{we can add } n \text{ to any part} \\ (\text{hence, } k \cdot |A(n-1,k)|).}} \qquad (1)$$

Let $A_k(x) = \sum_{n \geq 0} \left\{ {n \atop k} \right\} x^n$. (It's better to cycle over n since there's $k \cdot \left\{ {n-1 \atop k} \right\}$ in (1)).

$(1) \Longrightarrow A_k(x) = x A_{k-1}(x) + kx A_k(x) \Longrightarrow$

$$\Longrightarrow A_k(x) = \frac{x^k}{(1-kx)\cdots(1-x)} \underset{\substack{\text{partial} \\ \text{fractions}}}{=\!=} x^k \sum_{r=1}^{k} (-1)^{k-r} \frac{r^k}{r!\,(k-r)!} \frac{1}{1-rx}$$

Hence, $$\left\{ {n \atop k} \right\} = [x^n] A_k(x) = \frac{1}{k!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} r^n$$

$\rightsquigarrow k! \left\{ {n \atop k} \right\} = |\{ f : [n] \to [k] \text{ surjective}\}|$. Indeed, each part can be linked bijectively to a number j in $[k]$; hence to get all surjective functions we need to multiply by $k!$.

Exponential generating function (e.g.f)

$$\{a_n\}_{n \in \mathbb{N}} \xleftrightarrow{\text{e.g.f}} f(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n$$

$$\{a_{n+h}\}_{n \in \mathbb{N}} \longleftrightarrow D^h f \quad (\text{shifting})$$

Exercise Let's use e.g.f's to find a formula for F_n's.

$$F_{n+2} = F_{n+1} + F_n \implies$$

$$\implies F'' = F' + F. \qquad x^2 = x + 1 \implies$$

$$\implies \begin{cases} r_+ = (1+\sqrt{5})/2 = \varphi \\ r_- = (1-\sqrt{5})/2 \end{cases}$$

Hence $F(x) = \lambda_1 e^{r_+ t} + \lambda_2 e^{r_- t}$.

$F_0 = 0, \; F_1 = 1 \implies$

$$\implies \begin{cases} \lambda_1 + \lambda_2 = 0 & (F_0) \\ \lambda_1 r_+ + \lambda_2 r_- = 1 & (F_1) \end{cases} \implies \begin{cases} \lambda_2 = -\lambda_1 \\ \lambda_1 r_+ - \lambda_1 r_- = 1 \end{cases}$$

$$\rightsquigarrow \lambda_1 = \frac{1}{r_+ - r_-} = \frac{1}{\sqrt{5}} \implies \lambda_2 = -\frac{1}{\sqrt{5}}.$$

Therefore, $F(x) = \frac{1}{\sqrt{5}} \left(e^{r_+ t} - e^{r_- t} \right) \implies$

$$\implies \boxed{F_n = \frac{1}{\sqrt{5}} \left(r_+^{\,n} - r_-^{\,n} \right)} \quad \checkmark$$

- $\{P(m)\, a_m\} \xleftrightarrow{e.g.f.} P(xD)\, f$ (again) $\swarrow$ (product)
- $f \cdot g \xleftrightarrow{e.g.f.} \left\{ \sum_{k=0}^{m} \frac{a_k}{k!} \frac{b_{m-k}}{(m-k)!} m! = \sum_{k=0}^{m} \binom{m}{k} a_k b_{m-k} \right\}$

Example (Bell numbers)

$B_0 = 1, \quad B_m \triangleq \#\{\text{set part. of } [m]\} = \sum_{k=0}^{m} {m \brace k}.$

Let $B(x)$ be $\{B_m\}$'s e.g.f. Since $B_{m+1} = \underbrace{\sum_{k=0}^{m} \binom{m}{k} B_{m-k}}$,

Then:

cycling over how many el. are in the subset of $m+1 \in [m+1]$.

$$B'(x) = e^x \cdot B(x) \implies$$

$$\implies B(x) = e^{e^x - 1}.$$

Example (Derangements)

$D_m \triangleq \#\{\text{perm.} \in S_m \text{ with no fixed points}\}$

$D_0 \triangleq 1, \quad D_1 = 0$. Then $|S_m| = m! = \sum_{k=0}^{m} \underbrace{\binom{m}{k}}_{\text{fixed points}} D_{m-k}$ (2)

$D(x) \triangleq \sum_{i \geq 0} D_i / i! \; x^i$ (e.g.f.)

From (2) we get $D(x) \cdot e^x = \frac{1}{1-x} \implies$

$$\implies D(x) = \frac{e^{-x}}{1-x} \implies D_m = m! \Bigg[\underbrace{\sum_{k=0}^{m} \frac{(-1)^k}{k!}}_{\sim e^{-1}} \Bigg] \quad \left(\implies D_m = \left\lfloor \frac{m!}{e} \right\rceil \right).$$

$\Downarrow$

$$D(x) = x D(x) + e^{-x} \implies D_m = m \cdot D_{m-1} + (-1)^m.$$