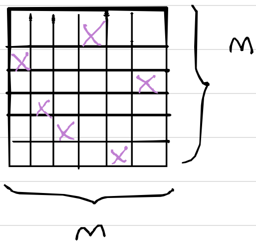
# Applications of the generalised incl.-excl. principle




$(1\ 5)(2\ 3)(4\ 6) \in S_6$

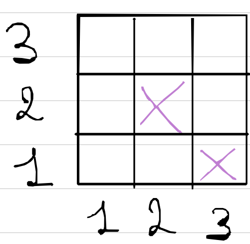
Let's consider an $m \times m$ board. We can study the non attacking rook's conf.'s on it, where a rook is a piece that is allowed to move only along the vertical line or the horizontal one (just like chess).

There cannot be more than $m$ non attacking rooks by the pigeonhole principle.

Non attacking means:

- each row contains at most one rook
- " column "

Hence we can identify such configurations with the injective partial functions from $[m]$ to itself, where the columns behave as the domain and the rows as the codomain, with the $\uparrow\!\!\rightarrow$ axes convention:




$$\longleftrightarrow \qquad f(m) = \begin{cases} \backslash & m = 1 \\ 2 & m = 2 \\ 1 & m = 3 \end{cases}$$

Hence the conf's of $m$ non-attacking rooks identify with the permutations in $S_m$.

Let's now consider $A \subseteq \underbrace{B}_{\text{board}} \triangleq [m]^2$.

We define $r_k(A) \triangleq \#\{$ conf's on $k$-non attacking rooks in $A\}$.

$\rightsquigarrow$ $r_m(B) = m!$, $r_1(A) = |A|$, $r_k(A) = 0 \quad \forall k > m$.

Let $R(x, A) \triangleq \sum_{k=0}^{m} r_k(A)\, x^k$ be the o.g.f. for $\{r_k(A)\}_k$, called the rook polynomial for $A$.

Let $H(x, A) \triangleq \sum_{\substack{(\sigma \in S^n) \\ \sigma \text{ conf. for} \\ m \text{ non att. rooks}}} t^{|\sigma \cap A|}$, called the hit polynomial for $A$ (in $B$).

**Theorem** $H(t, A) = \sum_{k=0}^{m} (m-k)!\, r_k(A)\, (t-1)^k$. (1)

Let $A_i = \{$ conf's hitting $A$ by the $i$-th column$\}$.

Then:
$$\sum_{k=0}^{m} e_k t^k = \underbrace{\sum_{k=0}^{m} a_k (t-1)}$$

$H(t, A)$, since $e_k = \#\{$ conf's hitting $A$ in exactly $k$ columns$\}$

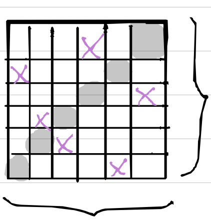
$a_k = \#\{$ conf's hitting $A$ in at least $k$ columns$\} =$

$= \underbrace{r_k(A)}_{\text{we choose a conf. in } A} \underbrace{(m-k)!}_{\text{we are free to choose the rest of the conf}}$

Hence $H(t, A) = \sum_{k=0}^{m} (m-k)!\, r_k(A)\, (t-1)^k$. $\square$

**Example** (derangements)

Derangements identify with conf's in $B$ avoiding the diagonal. Let $A$ be such diagonal of the board. We get, by the previous Theorem:



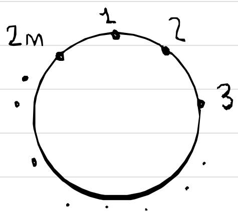
$m$

$m$

$$|D_m| = H(0, A) = \sum_{k=0}^{m} (m-k)!\, \underbrace{r_k(A)}_{\binom{m}{k}}\, (-1)^k =$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)!\,.$$

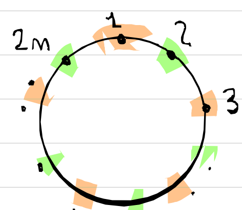
# The ménage problem




Let's consider a round table with 2n seats. Suppose we have a set of n husbands $H_i$ and another one of n wives $W_i$, where $(H_i, W_i)$ is a couple. We want to count how conf's of the table are s.t.

- husbands and wives alternate
- nobody sits next to his or her partner

Let's use $M_n$ to indicate the number of such conf's.

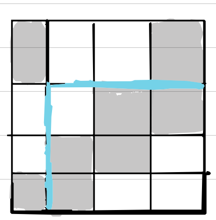



Let's fix the husbands. A table might start with a man or a woman, hence husbands will be either in or (2 choices); we can choose $n!$ perm's for the husbands. Hence:

$$M_n = 2 \cdot n! \cdot \#\{\text{possible conf's of the wives}\}$$

Wlog, husbands are in and are sat in order $(H_1, \dots, H_n)$. The possible conf's of the wives are the $\sigma \in S_n$ s.t. $\sigma(1) \notin \{1, n\}$, $\sigma(2) \in \{1, 2\}$, ...

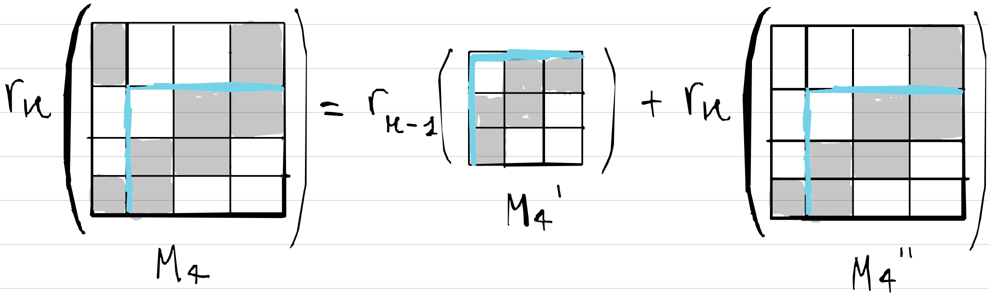
In a board this corresponds to counting n non-attack. rooks' conf's in B avoiding A = , denoted as $M_n$.

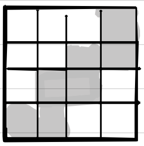


Ménage board $M_4$

Hence $\sum_{k=0}^{n} (-1)^k (n-k)!\, r_k(A)$.

Let's compute $r_k(A) = r_k(M_n)$. We notice there's an upper-left element in $M_n$, which we call $c_n$. We can either choose $c_n$ and reduce to counting rook conf's on $M_n'$ (obtained by removing 1st row and column); or we don't choose it, ignoring it on $M_n''$





$S_4$

Let $S_m$ be the staircase board in $[m]^2$ with subset $\{(1,1),(2,1),(2,2),\dots,(k,k-1),(k,k),\dots,(m,m-1),(m,m)\}$. It has $2(m-1)+1 = 2m-1$ elements.

- $M_m'$ is $S_{m-1}$ rotated by $180°$ degrees
- $M_m''$ is exactly $S_m$.

Hence we're left with computing $r_k(S_m)$. The only condition for a rook conf to be valid in $S_m$ is that it must not have rooks in two adjacent cells of the subset, hence we're counting combinations of $k$ elements in $2m-1$ places with no consecutive choices.

We'd have $2m-1-k$ gaps; if each gap is a vertical line (|), we're choosing the positioning of $k$ elements s.t. each element is before the first or last line or is between two lines; since there are $(2m-1-k)+1 =$
$= 2m-k$ positions and $k$ elements, we have:

$$r_k(S_m) = \binom{2m-k}{k}$$

Hence $r_k(M_m) = r_{k-1}(S_{m-1}) + r_k(S_m) = \binom{2(m-1)-(k-1)}{k-1} + \binom{2m-k}{k} =$

$$= \binom{2m-k-1}{k-1} + \binom{2m-k}{k} = \left(\frac{k}{2m-k}+1\right)\binom{2m-k}{k} = \frac{2m}{2m-k}\binom{2m-k}{k}.$$

**Theorem** (Touchard) The following formula holds:

$$\boxed{M_m = 2 \cdot m! \cdot \sum_{k=0}^{m} (-1)^k (m-k)! \cdot \frac{2m}{2m-k} \binom{2m-k}{k}}$$

Trivial by the previous considerations. $\square$

## Ferrers boards

We say an $m$-board is a Ferrers board if its subset consists of vertical piles of cells in weakly increasing order from left to right. Each Ferrers board is identified by a $m$-tuple $(a_1, \dots, a_m)$ with $0 \le a_1 \le \dots \le a_m$, where each $a_i$ represents how many cells are in the $i$-th column.

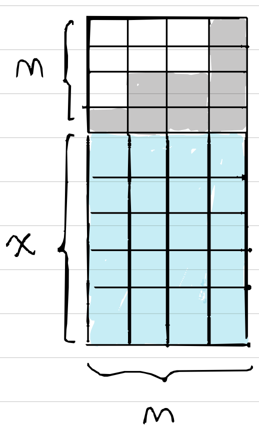

$(1,2,2,4)$

**Theorem** (Goldman-Joichi-White) Let $a_i$ be s.t. $0 \le a_1 \le \dots \le a_m$. Then the following formula holds:

$$\boxed{\prod_{i=1}^{m} (x + a_i - i + 1) = \sum_{k=0}^{m} r_{m-k}(F)\,(x)_k,}$$

where $F$ is the Ferrers board identified by $(a_1, \dots, a_m)$.

We prove the equality by double counting for $x > m$ (then the diff of the two polynomials will have infinite roots, hence they will be equal).

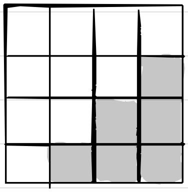



Let's add $x$ rows below $F$. We can count the conf's of $m$-non att. rooks in $F+x$ in two ways:

- $\prod_{i=1}^{m} (x + a_i - i + 1)$: for each column $i$ we have $x + a_i$ choices in the pile, but $i-1$ rows were already chosen.

- $\sum_{k=0}^{m} r_{m-k}(F)\,(x)_k$: we choose a conf. in $F$ with $m-k$ rooks, then we just need $k$ more rooks in $x$, for which we have $(x)_k$ choices. $\square$

Let $SS_n$ be the solid staircase board, i.e. the Ferrers board $(0,1,\dots,n-1)$.


$SS_4$

From the previous theorem we get:

$$x^n = \sum_{k=0}^{n} r_{n-k}(SS_n)\,(x)_k.$$

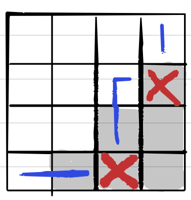
We also already knew that:

$$\underbrace{x^n}_{\substack{\text{func. from}\\ [n] \text{ to } [x]}} = \sum_{k=0}^{n} \underbrace{\left\{ {n \atop k} \right\}}_{\substack{\text{pre-images}\\ \text{with } k\\ \text{images}}} \underbrace{(x)_k}_{k \text{ images}}.$$

Since $\{(x)_k\}$ forms a basis, we instantly get:

$$\boxed{r_k(SS_n) = \left\{ {n \atop n-k} \right\}} \quad (2)$$

$\rightsquigarrow$ a bijection between $k$-partitions of $n$ and $k$-conf's of $SS_n$ is given as follows:


$\{\{1,3,4\},\{2\}\}$

- given a $k$-conf in $SS_n$, we give the following relation:

  $i \sim j \iff \exists$ X s.t. $j$ is above X and $i$ on its left or viceversa

  The quotient by the transitive and reflexive closure of such relation gives the desired partition;

- there's an inverse process to the operation done above.

# Eulerian polynomials

Let $\sigma \in S_n$. We say $i \in [n]$ is a rise (strict rise) of $\sigma$ if $\sigma(i) \geq i$ ($\sigma(i) > i$); we say $i$ is a fall if $\sigma(i) < i$ (i.e., if it's not a rise).

Let $\text{rise}(\sigma) \triangleq \# \{ i \mid i \text{ rise for } \sigma \}$, $\text{srise}(\sigma) \triangleq \{ \text{"} \mid \text{" strict "} \}$ and $\text{fall}(\sigma) \triangleq \# \{ \text{"} \mid \text{" fall "} \}$.

$\rightsquigarrow$ $\text{rise}(\sigma) + \text{fall}(\sigma) = n$

We say $E_n(t) \triangleq \sum_{\sigma \in S_n} t^{\text{rise}(\sigma)}$ is the $n$-th Eulerian polynomial, whose coeff's count permutations with a fixed number of rises.

$\rightsquigarrow$ The reciprocal polynomial of $E_n(t)$, i.e., $t^n E(1/t)$ counts falls:

$$t^n E_n(1/t) = \sum_{\sigma \in S_n} t^{\text{fall}(\sigma)}$$

$E_n(t)$ is exactly the hit polynomial of $\overline{SS_n}$ (the compl. of $SS_n$):

$$E_n(t) = H(t, \overline{SS_n}),$$

hence $t^n E_n(1/t) = H(t, SS_n)^*$. By rotating $SS_n$ by $180°$, we get the board whose hit polynom. is $\sum_{\sigma \in S_n} t^{\text{srise}(\sigma)}$. Hence, since this has the same hit polynomial of $SS_n$, we get:

$$\sum_{\sigma \in S_n} t^{\text{srise}(\sigma)} = \sum_{\sigma \in S_n} t^{\text{fall}(\sigma)} = t^n E_n(1/t) = H(t, SS_n) \quad (3)$$

* in general $t^n H(1/t, A) = H(t, A)$.

$\rightsquigarrow$ There always exists a rise in $\sigma \in S_n$; otherwise we'd have $\sigma(i) < i \ \forall i \Longrightarrow$

$\Longrightarrow i > \sigma(i) > \sigma^2(i) > \dots > \sigma^n(i)$. By the pigeonhole principle, there's a $1 < j \le n$ s.t. $\sigma^j(i) = i$, hence we'd have $i > i$, $\lightning$.

Therefore $t \mid E_n(t)$

Let $\sigma^*$ be $\sigma + 1$ (i.e., $\sigma^*(i) =$ "$\sigma(i) + 1 \bmod n$" $= \begin{cases} 1 & \text{if } \sigma(i)+1 = n+1 \\ \sigma(i)+1 & \text{otherwise} \end{cases}$). We observe that:

$$\operatorname{srise}(\sigma^*) = \operatorname{rise}(\sigma) - 1,$$

Therefore, by (1), (2) and (3) ($\cdot^*$ is invertible):

$$H(t, SS_n) = \sum_{\sigma^* \in S_n} t^{\operatorname{srise}(\sigma^*)} = \sum_{\sigma \in S_n} t^{\operatorname{rise}(\sigma) - 1} =$$

$$= E_n(t) / t \Longrightarrow$$

$$\Longrightarrow \boxed{E_n(t) = t\, H(t, SS_n) = t \sum_{k=0}^{n} (n-k)! \left\{ {n \atop n-k} \right\} (t-1)^k}$$