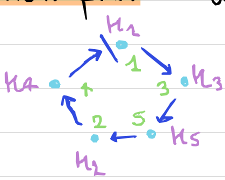
The exponential formula

We want to find a general construction for certain labelled objects in order to apply a unified formula for all of them.

Def. A **CARD** is a pair (S, p) where p is a "picture" (e.g., a structure with empty labels) and $S \subseteq \mathbb{N}$ is a finite $\underbrace{\text{set of labels}}_{\text{ordered with the usual } \leq}$. We $|S|$ its **WEIGHT**.

Example. We can represent an m-cycle as a labelled conf. in $[m]$ to which we can associate a set $S = \{k_1 \leq \dots \leq k_m\}$ of labels.



$(1,3,5,2,4)$ with labels $k_1 \leq \dots \leq k_5$ is $(k_1, k_3, k_5, k_2, k_4)$. (just like $(1,3,5,2,4)$) a "prototype"

Def. A card (S, p) is said to be **STANDARD** is $S = [|S|]$.

Def. A **RELABELLING** of a card (S, p) is a card (S', p) with $|S'| = |S|$. We say (S', p) is a **STANDARD RELABELLING** if (S', p) is a standard card.

Def. A **HAND** is a set of cards whose labels form a partition of $[m]$ for some $m \in \mathbb{N}^+$. We call the sum of the weights of its cards its **WEIGHT**.

Example. We can think of a permutation $\sigma \in S_m$ as a hand of cycles, which behave as its cards.

$$(1,6,2)(3,4)(5) \in S_6$$

$$(\{1 \leq 2 \leq 6\}, (1,3,2)) \qquad (\{3 \leq 4\}, (1,2)) \qquad (\{5\}, (1))$$

Def. A **DECK** is a finite set of $\underline{\text{standard}}$ cards with the same weight; we call such $\underline{\text{weight}}$ the **WEIGHT** of the deck.

$\rightsquigarrow$ a deck contains the "prototype of the cards", i.e., "their configurations".

Def An **EXPONENTIAL FAMILY** $\mathcal{F}$ is a collection of decks $\{D_i\}_{i\in\mathbb{N}}$ s.t. D_i has weight i and $D_0 \triangleq 0$.

If we set $d_i \triangleq |D_i|$, we can consider the e.g.f. of $\{d_i\}$:

$$\boxed{D(x) \triangleq \sum_{m\geq 0} \frac{d_m}{m!} x^m,}$$

which is called the **DECK ENUMARATOR** of $\mathcal{F}$.

Example (permutations) As we saw, permutations are hands made of cycles. Then, a suitable family for permutations is $\{D_i\}_{i\in\mathbb{N}}$ s.t.

$$D_i \triangleq \{i\text{-cycles in } S_i\} \Longrightarrow d_i = \frac{i!}{i} = (i-1)! \qquad (i\neq 0)$$

Hence $D(x) = \sum_{i\geq 1} \frac{(i-1)!}{i!} x^i = \sum_{i\geq 1} \frac{x^i}{i} = -\log(1-x) = \log\left(\frac{1}{1-x}\right)$.

Def For each $m, k \geq 0$ we let $h(m,k)$ be the **number of hands of weight m that consist of k cards** s.t. each card is a relabelling of some card in some deck of $\mathcal{F}$. By convention $h(m,0) = \delta_{m,0}$.

We call $H(x,y) \triangleq \sum_{m,k\geq 0} \frac{h(m,k)}{m!} x^m y^k$ the **HAND NUMERATOR** of $\mathcal{F}$.

Example (permutations) $h(m,k)$ counts in $\mathcal{F}_a$ the number of permutations $\in S_a$ with k cycles.

$\rightsquigarrow$ by setting $y=1$ in $H(x,y)$ one gets the e.g.f. of $\{h(x)\}$ where:

$$h(m) \triangleq \#\{\text{hands of weight } m\}.$$

Our goal is to find a relation between $D(x)$ and $H(x,y)$.

Before that we introduce such a relation, we explore the direct sum operation.

Def. Let $F = \{D_i\}_{i\in\mathbb{N}}$ and $F' = \{D_i'\}_{i\in\mathbb{N}}$ be two exponential families s.t. $\underbrace{D_i \cap D_i' = \emptyset}_{\text{all pictures are diff.}}$ $\forall i$. Then we define their **MERGE** as:

$$F'' = F \oplus F' = \{D_i'' = D_i \sqcup D_i'\}_{i\in\mathbb{N}}.$$

$$\rightsquigarrow D_{F\oplus F'}(x) = D_F(x) + D_{F'}(x).$$

Prop. (fundamental lemma of labelled counting) If $F'' = F \oplus F'$, then:

$$\boxed{H_{F''}(x,y) = H_F(x,y) \cdot H_{F'}(x,y)}$$

We observe that a hand of weight n made of k cards can be split in two hands: one coming from F and one coming from F' s.t. their weights sum to n and the numbers of their cards sum to k.

" of weight $3+5+6=14$
with $2+1=3$ cards in F''

hand of weight $3+5=8$ with 2 cards in F

" 6
" 1 card in F'

Hence the following holds:

we choose the labels we pick in F

$$\boxed{h_{F''}(n,k) = \sum_{n'=0}^{n} \sum_{k'=0}^{k} \binom{n}{n'} h_F(n',k')\, h_{F'}(n-n', k-k')}$$

Since the RHS is exactly $\left[\frac{x^n}{n!} y^k\right] H_F(n,k) H_{F'}(n,k)$, the theorem is proved. $\square$

Theorem (Exponential formula) The following formula holds:

$$\boxed{H(x,y) = e^{y\,D(x)}}$$

Let F_r be an exponential family s.t. $D_i = \emptyset \Leftarrow i \neq r$ and D_r consists of only one card.

Then $D_{F_r}(x) = \dfrac{x^r}{r!}$. Let's count now $h(m,k)$: if $m \neq k \cdot r$, $h_{F_r}(m,k)$ will necessarily be zero since there is no way to build a hand of weight m; if $m = k \cdot r$, we must employ r cards and we can do so in $\dfrac{1}{k!}\binom{k \cdot r}{\underbrace{r\, r \cdots r}_{k \text{ times}}}$ ways

To avoid permutations

Let $\binom{m}{m_1 \cdots m_k}$ denote the MULTINOMIAL COEFFICIENT which counts how many families $\{S_i\}_{i=1-k}$ we can have s.t. $S_i \subseteq [m]$, $|S_i| = m_i$, $S_i \cap S_j = \emptyset \Leftarrow i \neq j$, when $m_1 + \ldots + m_k = m$.

We observe that $\binom{m}{m_1 \cdots m_k} = \binom{m}{m_1}\binom{m-m_1}{m_2} \cdots \binom{m-m_1-\cdots-m_{k-1}}{m_k}$, hence:

$$\boxed{\binom{m}{m_1 \cdots m_k} = \frac{m!}{m_1! \cdots m_k!}}$$

$$\longrightarrow \binom{m}{k\ \ m-k} = \binom{m}{k}.$$

Therefore $h_{F_r}(kr, k) = \dfrac{(kr)!}{k!\,r!^k} \Longrightarrow H_{F_r}(x,y) = \displaystyle\sum_{k \geq 0} \frac{\cancel{(kr)!}}{r!^k} \frac{x^{kr}}{\cancel{(kr)!}}\, y^k/k! =$

$$= \sum_{k \geq 0} \left(\frac{x^r}{r!}\, y\right)^k / k! = e^{y \frac{x^r}{r!}}$$

Since each exponential family is the (possibly infinite) merge of families isomorphic to F_r, by applying the previous lemma the proof follows. □

$H = \prod_i H_i$ + $e^{x+y} = e^x e^y$

$\rightsquigarrow$ since $H(x,y) = e^{y D(x)}$, $\boxed{h(m,k) = \left[\frac{x^m}{m!}\right] \frac{D(x)^k}{k!}}$.

Example. (permutations) Let's count how many permutations in S_m have k cycles:

$$D(x) = \log\left((1-x)^{-1}\right) \xRightarrow[H(x,y)=(1-x)^{-y}]{\text{exp. formula}} h(m,k) = \left[\frac{x^m}{m!}\right] \frac{\log\left(\frac{1}{1-x}\right)^k}{k!}.$$

$h(m,k)$ is denoted as $\left[{m \atop k}\right]$ and such numbers are called the (unsigned) Stirling numbers of first kind.

$$\sum_{k\geq 0} \left[{m \atop k}\right] y^k = \left[\frac{x^m}{m!}\right] (1-x)^{-y} = m!\,(-1)^m \binom{-y}{m} = (-1)^m (-y)_m =$$

$$= \underbrace{(-1)^m \, (-1)^m \, (y+m-1)_m}_{} = (y+m-1)_m \triangleq \underbrace{y^{(m)}}_{\text{rising factorial}} \quad (1)$$

$(-y)_m = (-y)\cdots(-y-m+1)$
$= (-1)^m (y+m-1)\cdots y =$
$= (-1)^m (y+m-1)_m =$
$= (-1)^m y^{(m)}$

By substituting $y = -x$ in (1) we get:

$$(x)_m = (-1)^m (-x)^{(m)} = \sum_{k\geq 0} (-1)^{m-k} \left[{m \atop k}\right] x^k. \quad (2)$$

Therefore:

- The coeff.'s of $x^{(m)}$ are the (unsigned) Stirling numbers of first kind $\left[{m \atop k}\right]$
- The coeff.'s of $(x)_m$ are the (signed) Stirling numbers of first kind (see (2))

Moreover (2) gives us the elements of the matrix changing:

$$\underbrace{\{1, (x)_1, (x)_2, \dots\}}_{\text{falling factorial basis}} \quad \text{into} \quad \underbrace{\{1, x, x^2, \dots\}}_{\text{monomial basis}}$$

Example. (partitions) Let's count partitions. Each D_i will have only one card, i.e., $D_i = \{[i]\}$.

Hence $D(x) = \sum_{k \geq 1} x^k/k! = e^x - 1 \Rightarrow H(x,y) = e^{y(e^x-1)}$.

Recall that $\left\{ {m \atop k} \right\} \triangleq h(m,k)$. Then:

$$\sum_m \left\{ {m \atop k} \right\} \frac{x^m}{m!} = [y^k] H(x,y) = \frac{D(x)^k}{k!} = \frac{(e^x-1)^k}{k!},$$

which gives us the e.g.f. for the Stirling numbers of second kind.

Moreover — recalling that $b(m) \triangleq \sum_k \left\{ {m \atop k} \right\}$ —:

$$B(x) = H(x,1) = e^{e^x-1},$$

which we already knew.

Example. Let $a(m) = \# \{\sigma \in S_m \mid \sigma^{m} = 1\}$, for a fixed $m \in \mathbb{N}$. Recall that $\sigma^m = 1$ iff every cycle in σ has a length that divides m.

We can consider a family where $d_i = (i-1)!$ if $i \mid m$ and $d_i = 0$ otherwise.

Then $\underbrace{H(x) = H(x,1)}_{\sum_m \frac{a(m)}{m!} x^m} = e^{D(x)}$, where $D(x) = \sum_{d \mid m} \frac{x^d}{d!}$.

For example, if $m=2$, $H_2(x) = e^{x + \frac{x^2}{2}}$.

$\leadsto$ such an example can be adapted to find the number of permutations $\in S_m$ with odd cycles.

In all the examples we gave above, we computed $D(x)$ first and then we deduced $H(x,y)$. Sometimes it's better to go in the other direction.

$$H(x,y) = e^{yD(x)} \implies \frac{\partial}{\partial x} H(x,y) = y\, D'(x)\, e^{yD(x)}.$$

Recall that deriving an e.g.f. is equivalent to shifting by one its coeff. Thus:

$$h(m+1, k) = \left[\tfrac{x^m}{m!}\right]\left[y^k\right] D'(x) \cdot y e^{yD(x)}$$

$$= \sum_j \binom{m}{j} d_{m+1-j}\, h(j, k-1).$$

By summing over all k's (with $h(m) \triangleq \sum_k h(m,k)$):

$$h(m+1) = \sum_j \binom{m}{j} d_{m+1-j}\, h(j) \implies$$

$$\implies d_{m+1} = h(m+1) - \sum_{j \geq 1} d_{m+1-j}\, h(j), \qquad (3)$$

which gives us a tool to get $D(x)$ from $H(x,1) = H(x)$.

Example. (connected labelled graphs) We know how many graphs on m vertices there are:

$$h(m) = 2^{\binom{m}{2}}$$ for each edge, we need to choose whether to include it or not

Hence the number of connected labelled graphs (which make up for our decks) is given by (3):

$$d_{m+1} = 2^{\binom{m+1}{2}} - \sum_{j \geq 1} 2^{\binom{m+1-j}{2}} d_{m+1-j}.$$