Lagrange inversion

We're interested in finding a formula for the coefficients of polynomials arising from inverse problems.

For example, we want to solve $f(g(x)) = x$ for g, i.e., we want a formula for the coefficients of g, the compositional inverse $f^{\langle -1 \rangle}(x)$.

Notice that $f(g(x)) = x \iff \dfrac{x}{f(g(x))} = 1 \iff g(x) = x \dfrac{g(x)}{f(g(x))}$
$\iff g(x) = x\, G(g(x))$ where $G(x) = x/f(x)$.

We'll state the inversion formula, which we will prove later on.

Theorem (Lagrange inversion) Let $\mathbb{K}$ be a field of characteristic 0. Let $G(x) \in \mathbb{K}[[x]]$ with $G(0) \in \mathbb{K}^*$ (i.e., $G(0) \neq 0$) and let $f(x) \in \mathbb{K}[[x]]$ be s.t.

$$f(x) = x\, G(f(x)).$$

Then:

$$m\,[x^m]\, f(x)^k = k\,[x^{m-k}]\, G(x)^m$$

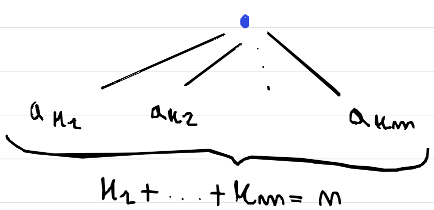
$\rightsquigarrow$ Therefore $[x^m]\, f^{\langle -1 \rangle}(x)^k = \dfrac{k}{m}[x^{m-k}]\, x^m \cdot f^{-m} = \dfrac{k}{m}[x^{-k}]\, f^{-m}(x)$.

Observe that $m\,[x^m]\, H(f(x)) = [x^{m-1}]\, H'(x)\, G(x)^m$ for any power series $H(x)$. Indeed, this is true for monomials thanks to Lagrange's inversion theorem and the general proof follows due to the linearity of $[x^m]$.

Example. (complete m-ary trees with n leaves) Let $m \geq 2$. We want to compute a_n, where

$$a_n \triangleq \# \{ \text{complete } m\text{-ary trees with } n \text{ leaves and labels in BF order} \}.$$

Notice that $a_1 = 1$. Moreover, each n-tree is made up of a k_1-subtree, a k_2-subtree, $\dots$, a k_m-subtree.



Hence, for $n \geq 2$:

$$a_n = \sum_{\substack{k_1 + \dots + k_m \\ = n}} a_{k_1} \cdots a_{k_m}$$

Let $A(x)$ be the o.g.f for $\{a_n\}_{n \in \mathbb{N}}$. Then, from $a_1 = 1$ and (2) we get:

$$A(x) = \overbrace{x}^{a_1} + \overbrace{A(x)^m - 0 \cdot x}^{a_n \ n \geq 2} = $$
$$= x + A(x)^m.$$

Thus, $A(x)\left(1 - A(x)^{m-1}\right) = x \implies A(x) = x \cdot \dfrac{1}{1 - A(x)^{m-1}} = x\, G(A(x))$,

where $G(x) = \dfrac{1}{1 - x^{m-1}}$.

We can then apply Lagrange's inversion to get:

$$a_n = [x^n]\, A(x) = \frac{1}{n} [x^{n-1}]\, G(x)^n = \frac{1}{n} [x^{n-1}] \left(1 - x^{m-1}\right)^{-n}$$
$$= \frac{1}{n} [x^{n-1}] \sum_k \binom{-n}{k} (-1)^k x^{km-k}$$

Therefore:

$$a_n = \begin{cases} 0 & \text{if } m-1 \nmid n-1 \\ \dfrac{1}{n} (-1)^k \dbinom{-n}{k} = \dfrac{1}{n} \dbinom{n+k-1}{k} & \text{if } n-1 = k(m-1) \end{cases}$$

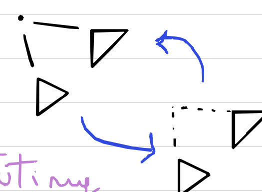
$\rightsquigarrow$ notice that — for $m=2$ — $a_n = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}$.

Indeed $A(x) = x + A(x)^2 \Rightarrow A(x) = \frac{-1-\sqrt{1-4x}}{2} = x\, C(x)$.

Example (labelled trees on n nodes) Let's compute the num. of labelled trees on n nodes, t_n'. Notice it's easier to count labelled rooted trees first (t_n) and then use:

$$t_n' = n \cdot t_n$$ (each node can be designed as the root)

Let f_n be the num. of labelled rooted forests on n vertices. Given a lab. roo. tree on $n+1$ vertices, we can remove the root, rescale the labels and get a family of trees (i.e., a forest); similarly we can merge a forest on n nodes to get a tree on $n+1$ nodes; rescaling gives us $n+1$ choices for the label of the new root, hence:



$$t_{n+1} = (n+1) f_n$$

$$\Longrightarrow T(x) = x \cdot F(x) \quad \text{\} by substituting}$$

By the exp. formula, $F(x) = e^{T(x)}$ ("forests $\longleftrightarrow$ hand of trees"), hence:

$$T(x) = x \cdot e^{T(x)} \rightsquigarrow G(x) = e^x.$$

Thus by Lagrange's inversion:

$$t_n = n!\,[x^n]\,T(x) = \frac{n!}{n}\,[x^{n-1}]\, \overbrace{e^{nx}}^{G(x)^n} =$$

$$= \frac{\cancel{n!}}{\cancel{n}} \cdot \frac{n^{n-1}}{\cancel{(n-1)!}} = n^{n-1} \xrightarrow{/n} t_n' = n^{n-2}.$$

Proving Lagrange's inversion theorem

→ let's prove the uniqueness of the solution first.

$$f(x) = x \cdot G(f(x)) \Longrightarrow$$

$$\Longrightarrow \sum_{i\geq 0} f_i x^i = x \cdot \sum_{i \geq 0} G_i \cdot \Big(\sum_{j\geq 0} f_j x^j\Big)^i \Longrightarrow$$

$$\overset{f_0=0}{\Longrightarrow} \sum_{i\geq 1} f_i x^i = x \cdot \sum_{i\geq 0} G_i \cdot \Big(\sum_{j\geq 1} f_j x^j\Big)^i \Longrightarrow$$

$$\overset{f_1=G_0}{\Longrightarrow} G_0 x + \sum_{i\geq 2} f_i x^i = x \cdot \sum_{i\geq 0} G_i \Big(G_0 x + \sum_{j\geq 2} f_j x^j\Big)^i$$

$$\overset{f_2=G_0^2}{\Longrightarrow} \ldots$$

One can prove by induction that f_m is determined by $\{f_i\}_{i<r}$ and $\{G_i\}_{i<r}$, hence $F(x)$ is unique.
Thus it suffices that $F(x)$ s.t. $[x^m]\, F(x) = \frac{1}{m}\, [x^{m-1}]\, G(x)^m$ is actually a solution for $f(x) = x\, G(f(x))$; then we'll prove the general formula.

Let $f_m = [x^m]\, f(x) \overset{!}{=} \frac{1}{m} [x^{m-1}]\, G(x)^m =$

$$= \frac{1}{m} \sum_{\substack{k_1+\ldots+\\ +k_m = m-1}} G_{k_1} \cdots G_{k_m} =$$

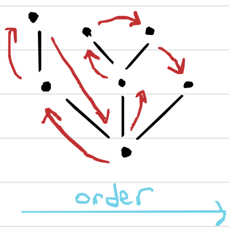
$$= \frac{1}{m} \sum_{\substack{p_0+\ldots+p_{m-1}=m \\ 0\cdot p_0 + 1\cdot p_1 + \\ +\ldots+(m-1)p_{m-1} = \\ = m-1}} G_0^{p_0}\, G_1^{p_1} \cdots G_{m-1}^{p_{m-1}} \qquad (1)$$

Our goal is to give a suitable comb. int. to (1).

We will give our proof through <u>planar trees</u>.

Def. A **PLANAR TREE** is a rooted tree with an ordered set of children. The degree of a node is the number of its children.

→ a planar tree $\overbrace{\text{can be translated}}^{w \triangleq}$ to a word in $\Sigma = \{x_i\}_{i \in \mathbb{N}}$ through the DFO (depth-first order):



$\xrightarrow{w}$ $x_3 x_1 x_0 x_2 x_0 x_0 x_0$ where $i \triangleq \deg(\text{current node})$

Such a map is <u>not</u> surjective on Σ^* (e.g., $x_0 x_1$ does <u>not</u> represent any tree).

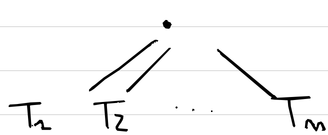
A necessary condition for $x_{k_1} \dots x_{k_m}$ to be an image is:

$$\underbrace{k_1 + \dots + k_m}_{\text{all children}} = \underbrace{m - 1}_{\text{all nodes except the root}}$$

→ which is the only non-child node

Yet it's <u>not</u> sufficient ($x_0 x_1$ is still a counterexample).

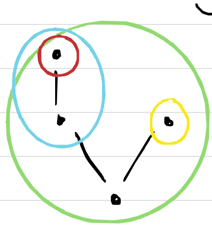
Notice such a word can be defined recursively:



$\xrightarrow{w}$ $x_m\, w(T_1)\, w(T_2) \dots w(T_m)$.

⤳ note that we can associate to a word in Σ a **Lukasiewicz path** s.t. $x_i \longleftrightarrow (1, i-1)$.

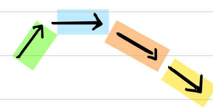
A word in Σ always stays above $y=0$ (except for the last entry) if seen as a path.



Tree

$\longrightarrow$ $x_2\, x_1\, x_0\, x_0$ $\xleftrightarrow{\text{bijection}}$

word



path

The previous cond. translates to "ending the path at -1"

This translates to the following condition:

$$(k_1 - 1) + \dots + (k_i - 1) \geq 0 \qquad \forall\, i = 1 \text{ — } m-1$$

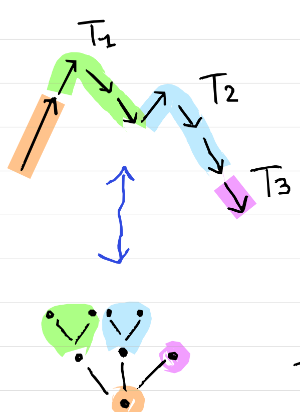
Theorem. A word in Σ is planar iff:

(i) $k_1 + \dots + k_m = m - 1$

(ii) $(k_1 - 1) + \dots + (k_i - 1) \geq 0 \qquad \forall\, i = 1 \text{ — } m-1$

($\Rightarrow$) seen.

($\Leftarrow$) let T be a tree defined s.t.



- The root has k_1 nodes
- The first of its k_1 trees is the corresponding tree of the path going from $(1, k_1 - 1)$ to the first node s.t. $y = (k_1 - 1) - 1$ (induction on length)
- analogous for the other paths (")

Thus $w(T) = x_{k_1} \dots$ $\square$

Let $F = \sum_{T \text{ pl. tree}} \underbrace{w(T)}_{\{x_i\} \text{ non-comm.!}}$. Then, by the seen factoriz. of $w(T)$ as $x_{k_1} w(T_1) \dots w(T_k)$:

$$F = \sum_{i \in \mathbb{N}} x_i F^i$$

Let φ be s.t. $x_i \overset{\varphi}{\longmapsto} G_i x$ (hence $\varphi(T) = x^{|V|-1}$) and call $f \triangleq \varphi(F)$; Then f satisfies:

$$f = \sum_{i \in \mathbb{N}} G_i \, x \, f^i =$$

$$= x \cdot G(f(x)),$$

where $G(x) = \sum_{i \in \mathbb{N}} G_i x^i$.

From the def. of F it's immediate to see that:

$$[x^m] f(x) = \sum_{\substack{p_0 + \dots + p_m = m \mid \text{nodes} \\ \text{edges } \{ 0p_0 + \dots + mp_m = m-1}} |T(p_0, p_1, \dots, p_m)| \, G_0^{p_0} \cdots G_m^{p_m} x^m.$$

Hence, in order to prove Lagrange's inversion Theorem for $k=1$ we must show that:

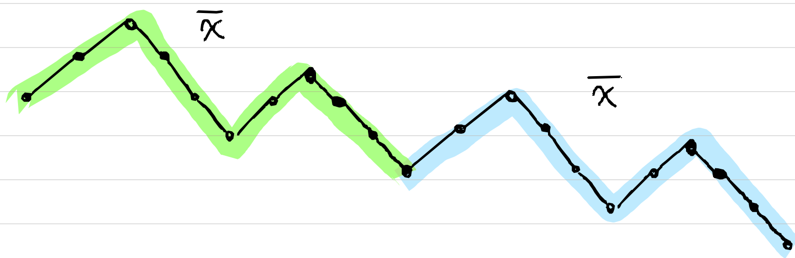
$$|T(p_0, \dots, p_m)| \overset{?}{=} \frac{1}{m} \binom{m}{p_0 \; \dots \; p_m}.$$

Consider $R(p_0, \dots, p_m) = \{ \text{rearrang. of } x_0^{p_0} \dots x_m^{p_m} \}$. It's Trivial to see That:

$$|R(p_0, \dots, p_m)| = \binom{m}{p_0 \; \dots \; p_m}.$$

Prop. $|R(p_0,\dots,p_m)| = m\,|T(p_0,\dots,p_m)|$ $\left(\begin{array}{l} p_0+\dots+p_m = m \\ 0\cdot p_0+\dots+m\cdot p_m = m-1\end{array}\right)$

Consider the Łukasiewicz path for $\bar{x} \in R(p_0,\dots,p_m)$:



rescaled!

Double the original path ( ) and consider paths with length m in it. These are exactly the cyclic perm. of $\bar{x}$.

Exactly one of such paths comes from a tree.

- Take the lowest point in ( ) as the starting point. That is a tree.
- Any other path necessarily goes below 0 at that point — thus cannot come from a tree.

The Theorem then follows naturally. $\square$

Therefore we've proved Lagrange's inv. Theo. in case $k=1$.

→ we can adapt the proof to $k \geq 2$ using forests with k planar trees.