APPLICAZIONI LINEARI

$\underline{v_1}, \dots, \underline{v_k} \in V$, $U = \mathrm{Span}(\underline{v_1}, \dots, \underline{v_k})$

Prop. $\underbrace{f(\underline{v_1}), \dots, f(\underline{v_k}) \text{ lin. ind.}}_{(1)} \iff \underline{v_1}, \dots, \underline{v_k}$ lin. ind. $\wedge$

$\wedge \quad \underbrace{U \cap \ker f = \{\underline{0}\}}_{(2)}$

(2) $\Rightarrow$ (1):

$$\alpha_1 f(\underline{v_1}) + \dots + \alpha_k f(\underline{v_k}) = \underline{0} \Rightarrow$$

$$\Rightarrow f(\underbrace{\alpha_1 \underline{v_1} + \dots + \alpha_k \underline{v_k}}_{\in U \cap \ker f}) = \underline{0} \Rightarrow$$

$$\Rightarrow \alpha_1 \underline{v_1} + \dots + \alpha_k \underline{v_k} = \underline{0} \Rightarrow \alpha_1 = \dots = \alpha_k = 0 \Rightarrow$$

$$\Rightarrow f(\underline{v_1}), \dots, f(\underline{v_k}) \text{ lin. ind.}$$

(1) $\Rightarrow$ (2):

$$\alpha_1 \underline{v_1} + \dots + \alpha_k \underline{v_k} = \underline{0} \Rightarrow$$

$$\Rightarrow \alpha_1 f(\underline{v_1}) + \dots + \alpha_k f(\underline{v_k}) = f(\underline{0}) = \underline{0} \Rightarrow$$

$$\Rightarrow \alpha_1 = \dots = \alpha_k = 0 \quad (\underline{v_1}, \dots, \underline{v_k} \text{ lin. ind.})$$

$\underline{v} \in U \cap \operatorname{Ker} f$

- $\underline{v} = \alpha_1 \underline{v_1} + \dots + \alpha_k \underline{v_k}$
- $f(\underline{v}) = \underline{0} \Rightarrow \alpha_1 f(\underline{v_1}) + \dots + \alpha_k f(\underline{v_k}) = \underline{0} \Rightarrow$

$\Rightarrow \alpha_1 = \dots = \alpha_k = 0 \quad (\underline{v} = \underline{0},\ U \cap \operatorname{Ker} f = \{\underline{0}\}).$

$\boxtimes$

Oss: $U \subset V$ ssp., $\dim U = k$, $\underline{v_1}, \dots, \underline{v_k}$ base di U

$f(U) = \operatorname{Span}(f(\underline{v_1}), \dots, f(\underline{v_k}))$

se $U \cap \operatorname{Ker} f = \{\underline{0}\}$:

- $\dim f(U) = k$

se f è iniettiva, $\operatorname{Ker} f = \{\underline{0}\}$:

- $\dim f(U) = k$ (in realta' anche per il caso infinito)
- f manda vett. lin. ind. in vett. lin. ind.

Oss. 2 $\dim f(U) = \dim U - \dim \operatorname{Ker} f|_U =$

$= \dim U - \dim(U \cap \operatorname{Ker} f).$

Oss. 3 $f: V \to W \wedge \dim W > \dim V \Rightarrow$

$\Rightarrow \dim W > \dim \text{Imm} f \Rightarrow f$ NON può essere surgettiva

Oss. 4 $f: V \to W \wedge \dim V > \dim W \Rightarrow$

$\Rightarrow \ldots$

Oss. 5 $f|_U : U \subset V \to W$, $u \mapsto f(u)$ è sempre ben definita e lineare, se f è lineare

$$\dim U = \dim \text{Ker} f|_U + \text{Im} f|_U = (*)$$

- $\text{Im} f|_U = f(U)$
- $\text{Ker} f|_U = U \cap \text{Ker} f$

$$(*) = \dim(U \cap \text{Ker} f) + \dim f(U)$$

Oss. 6 f isomorfismo $\Rightarrow \dim V = \dim W$

- $\text{Ker} f = \{\underline{0}\}$ perché f iniettiva
- $\dim W \underset{W = \text{Imm} f}{=} \dim \text{Imm} f = \dim V - \overbrace{\dim \text{Ker} f}^{0} = \dim V$

Matrici

$A = (A^1)(A^2)\cdots(A^n)$

$A\underline{x} = x_1 A^1 + \dots + x_n A^n \in \mathbb{K}^m$

$f_A : \mathbb{K}^n \to \mathbb{K}^m, \quad \underline{x} \mapsto A\underline{x}$

f_A è lineare:

- $f_A(\underline{x}+\underline{y}) = A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$
- $f_A(\alpha\underline{x}) = A(\alpha\underline{x}) = \alpha(A\underline{x}) =$
  $= \alpha f_A(\underline{x})$.

$\ker f_A = \left\{ \underline{x} \in \mathbb{K}^n \mid f(\underline{x}) = \underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{K}^m \right\}$,
ossia le soluzioni del sistema omogeneo
$A\underline{x} = \underline{0}$

$\mathrm{Imm}\, f_A = \left\{ f(\underline{x}) \in \mathbb{K}^m \mid \underline{x} \in \mathbb{K}^n \right\}$, ossia
$\mathrm{Span}(A^1, \dots, A^n)$, la cui dimensione è $rg(A)$.
Sapendo che un insieme di generatori di $\mathbb{K}^n$ genera $\mathbb{K}^m$ perché f_A è un app. lineare, la base canonica di $\mathbb{K}^n$ applicata genera $\mathbb{K}^m$.

es. $f: \mathbb{R}^3 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x-2y-z \\ 3x-y+3z \end{pmatrix} =$

$= x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} -2 \\ -1 \end{pmatrix} + z\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -2 & -1 \\ 3 & -1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\underline{x}}$, quindi

è app. lineare.

$\text{Imm}\, f_A = \text{Span}\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) =$

$= \text{Span}\Big(\underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}}_{\text{lin. ind.}}\Big) \Rightarrow$

$\Rightarrow \dim \text{Imm}\, f_A = 2 = \dim \mathbb{R}^2 \Rightarrow$

$\Rightarrow \mathbb{R}^2 = \text{Imm}\, f_A \Rightarrow f$ è surgettiva (i.e.

$A\underline{x} = \underline{b}$ ammette sempre soluzione).

$\dim \ker f_A = \dim \mathbb{R}^3 - \dim \text{Imm}\, f_A =$

$= 1$

Prop. $f: \mathbb{K}^n \to \mathbb{K}^m$ lineare $\iff \exists!\, A \in M(m,n,\mathbb{K}) \mid$

$f = f_A$

$$A = \Big(\overbrace{f(\underline{e}_1)}^{A^1} \Big| \overbrace{f(\underline{e}_2)}^{A^2} \Big| \dots \Big| \overbrace{f(\underline{e}_n)}^{A^n}\Big) \in M(m,n,\mathbb{K})$$

$x \in \mathbb{K}^n \qquad \underline{x} = x_1 \underline{e}_1 + \dots + x_n \underline{e}_n \Rightarrow$

$$\Longrightarrow f(\underline{x}) = x_1 f(\underline{e_1}) + \dots + x_n f(\underline{e_n}) =$$

$$= A\underline{x} \quad \forall\, \underline{x} \in \mathbb{K}^n$$

E' unicamente determinata la A:

- $A^i = f(e_i) = B^i \Longrightarrow A = B$

es. $f: \mathbb{R}^2 \to \mathbb{R}^3$ lineare: $f\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\0\end{pmatrix}$

$$f\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$\begin{cases} f(\underline{e_1}) - f(\underline{e_2}) = \begin{pmatrix}3\\1\\0\end{pmatrix} \\ 2f(\underline{e_1}) + f(\underline{e_2}) = \begin{pmatrix}1\\1\\-1\end{pmatrix} \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} f(\underline{e_1}) = \frac{1}{3}\left(\begin{pmatrix}3\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\-1\end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix}4\\2\\-1\end{pmatrix} \\ 2f(\underline{e_2}) = -\begin{pmatrix}3\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\-1\end{pmatrix} - f(\underline{e_1}) \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} f(\underline{e_1}) = \frac{1}{3}\begin{pmatrix}4\\2\\-1\end{pmatrix} \\ f(\underline{e_2}) = \frac{1}{3}\begin{pmatrix}-5\\-4\\-1\end{pmatrix} \end{cases}$$

Quindi $A = \left(f(\underline{e_1}) \mid f(\underline{e_2}) \right) =$

$$= \frac{1}{3} \underbrace{\begin{pmatrix} 4 & -5 \\ 2 & -4 \\ -1 & -1 \end{pmatrix}}_{\text{lin. ind.}} .$$

$\dim \operatorname{Imm} f_A = 2$

$\dim \ker f_A = 0 \Rightarrow$ iniettiva

es. $f : \mathbb{R}^3 \to \mathbb{R}^3$ $\qquad A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

- $\exists!\, U \subset \mathbb{R}^3$ ssp. $\mid \dim U = 1$ con f invariante (i.e. $f(U) \subset U$)
- $\exists!\, W \subset \mathbb{R}^3$ ssp. $\mid \dim W = 2$ con f invariante

$\exists \underset{\neq \underline{0}}{\underline{u_0}} \in \mathbb{R}^3 \mid U = \operatorname{Span}(\underline{u_0})$

- $f(\underline{u}) \subset \overbrace{\operatorname{Span}(\underline{u_0})}^{U} \iff f(\underline{u_0}) \subset \operatorname{Span}(\underline{u_0})$

$$f(\underline{u_0}) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ a+b-c \\ a+b+c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \longrightarrow$$

$$\Longrightarrow \begin{cases} 2a = \lambda a \\ a+b-c = \lambda b \\ a+b+c = \lambda c \end{cases} \Longrightarrow \begin{cases} (2-\lambda)a = 0 \\ a + (1-\lambda)b - c = 0 \\ a + b + (1-\lambda)c = 0 \end{cases} \Longrightarrow (\text{i}) \vee (\text{ii})$$

(i) $\Longrightarrow a=0 \quad \begin{cases} (1-\lambda)b-c=0 \Rightarrow c=(1-\lambda)b \\ b+(1-\lambda)c=0 \Rightarrow b+(1-\lambda)^2 b=0 \Rightarrow \end{cases}$

$$\Rightarrow \big(\underbrace{1+(1-\lambda)^2}_{>0}\big)\, b=0 \Rightarrow b=0 \Rightarrow c=0$$

$\Rightarrow u_0 = \underline{0}$ imp.

(ii) $\lambda=2 \quad \begin{cases} 0=0 \\ a-b-c=0 \\ a+b-c=0 \end{cases} \Rightarrow \begin{cases} b=0 \\ a=c \end{cases} \Rightarrow \underline{u} = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} =$

$$= a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underline{u_0} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ U=\mathrm{Span}(\underline{u_0})$$

$$U = \ker\big(f_{A-\underbrace{2}_{\lambda}I}\big) \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Riprendi le slide sull'imm. f-invariante