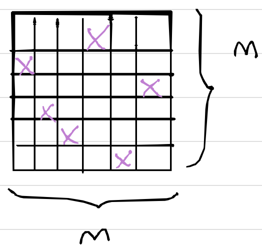
Applications of the generalised incl.-excl. principle



Let's consider an $n \times n$ board. We can study the non attacking rook's conf.'s on it, where a rook is a piece that is allowed to move only along the vertical line or the horizontal one (just like chess).

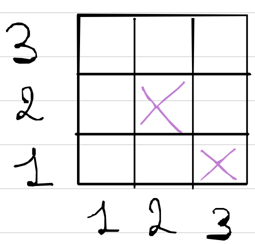
$(1\ 5)(2\ 3)(4\ 6) \in S_6$

There cannot be more than n non attacking rooks by the pigeonhole principle.

Non attacking means:

- each row contains at most one rook
- " column "

Hence we can identify such configurations with the injective partial functions from $[n]$ to itself, where the columns behave as the domain and the rows as the codomain, with the $\uparrow\!\rightarrow$ axes convention:



$\longleftrightarrow \quad f(n) = \begin{cases} \backslash & n=1 \\ 2 & n=2 \\ 1 & n=3 \end{cases}$

Hence the conf's of n non-attacking rooks identify with the permutations in S_n.

Let's now consider $A \subseteq \underbrace{B}_{\text{board}} \triangleq [m]^2$.

We define $r_k(A) \triangleq \#\{\text{conf's on } k\text{-non attacking rooks in } A\}$.

$\longrightarrow$ $r_m(B) = m!$, $r_1(A) = |A|$, $r_k(A) = 0 \quad \forall\, k > m$.

Let $R(x, A) \triangleq \sum_{k=0}^{m} r_k(A)\, x^k$ be the o.g.f. for $\{r_k(A)\}_k$, called the rook polynomial for A.

Let $H(x, A) \triangleq \sum_{\substack{(\sigma \in S^n) \\ \sigma \text{ conf. for} \\ m \text{ non att. rooks}}} t^{|\sigma \cap A|}$, called the hit polynomial for A (in B).

Theorem $\quad H(t, A) = \sum_{k=0}^{m} (m-k)!\, r_k(A)\, (t-1)^k. \qquad (1)$

Let $A_i = \{\text{conf's hitting } A \text{ by the } i\text{-th column}\}$.

Then:

$$\sum_{k=0}^{m} e_k t^k = \underbrace{\sum_{k=0}^{m} a_k (t-1)}$$

$H(t, A)$, since $e_k = \#\{\text{conf's hitting } A \text{ in exactly } k \text{ columns}\}$

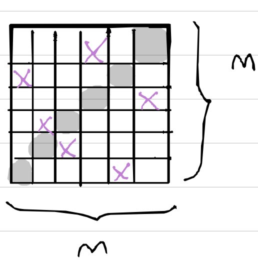
$a_k = \#\{\text{conf's hitting } A \text{ in at least } k \text{ columns}\} =$

$= \underbrace{r_k(A)}_{\substack{\text{we choose a} \\ \text{conf. in } A}} \; \underbrace{(m-k)!}_{\substack{\text{we are free} \\ \text{to choose the} \\ \text{rest of the conf.}}}$

Hence $\quad H(t, A) = \sum_{k=0}^{m} (m-k)!\, r_k(A)\, (t-1)^k.$ $\square$

Example (derangements)

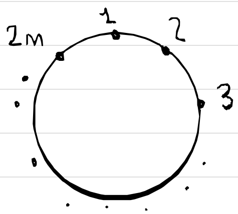
Derangements identify with conf's in B avoiding the diagonal. Let A be such diagonal of the board. We get, by the previous Theorem:



$$|D_m| = H(0, A) = \sum_{k=0}^{m} (m-k)!\, \underbrace{r_k(A)}_{\binom{m}{k}}\, (-1)^k =$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)!.$$

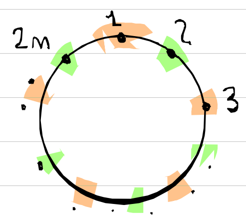
The ménage problem



Let's consider a round table with $2n$ seats. Suppose we have a set of n husbands H_i and another one of n wives W_i, where (H_i, W_i) is a couple. We want to count how conf's of the table are s.t.

- husbands and wives alternate
- nobody sits next to his or her partner

Let's use M_n to indicate the number of such conf's.

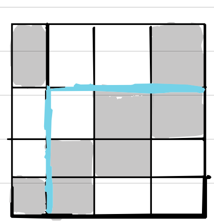


Let's fix the husbands. A table might start with a man or a woman, hence husbands will be either in or (2 choices); we can choose $n!$ perm.'s for the husbands. Hence:

$$M_n = 2 \cdot n! \; \# \{\text{possible conf's of the wives}\}$$

Wlog, husbands are in and are sat in order $(H_1, \dots, H_n)$. The possible conf's of the wives are the $\sigma \in S_n$ s.t. $\sigma(1) \notin \{1, n\}$, $\sigma(2) \notin \{1, 2\}$, ...

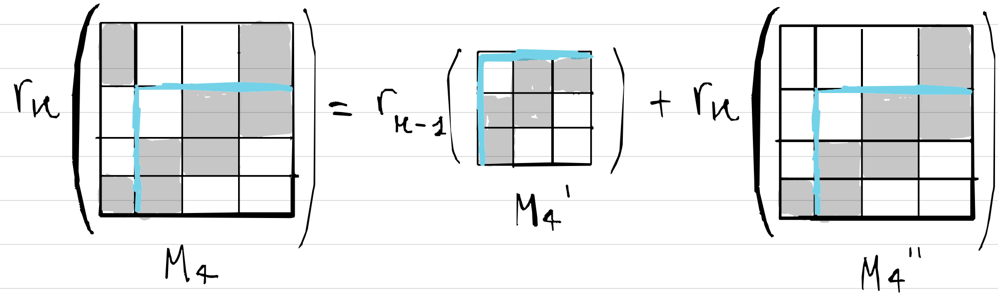
In a board this corresponds to counting n non-attack. rooks' conf's in B avoiding $A =$, denoted as M_n.

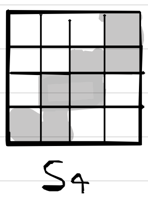


Ménage board M_4.

Hence $\sum_{k=0}^{n} (-1)^k (n-k)! \, r_k(A)$.

Let's compute $r_k(A) = r_k(M_n)$. We notice there's an upper-left element in M_n, which we call c_n. We can either choose c_n and reduce to counting rook conf's on M_n' (obtained by removing 1st row and column), or we don't choose it, ignoring it on M_n''





Let S_m be the staircase board in $[m]^2$ with subset $\{(1,1),(2,1),(2,2),\dots,(k,k-1),(k,k),\dots,(m,m-1),(m,m)\}$. It has $2(m-1)+1 = 2m-1$ elements.

- M_m' is S_{m-1} rotated by $180°$ degrees.
- M_m'' is exactly S_m.

Hence we're left with computing $r_k(S_m)$. The only condition for a rook conf. to be valid in S_m is that it must not have rooks in two adjacent cells of the subset, hence we're counting combinations of k elements in $2m-1$ places with no consecutive choices.

We'd have $2m-1-k$ gaps; if each gap is a vertical line (|), we're choosing the positioning of k elements s.t. each element is before the first or last line or is between two lines; since there are $(2m-1-k)+1 =$
$= 2m-k$ positions and k elements, we have:

$$\boxed{r_k(S_m) = \binom{2m-k}{k}}$$

Hence $r_k(M_m) = r_{k-1}(S_{m-1}) + r_k(S_m) = \binom{2(m-1)-(k-1)}{k-1} + \binom{2m-k}{k} =$

$$= \binom{2m-k-1}{k-1} + \binom{2m-k}{k} = \left(\frac{k}{2m-k}+1\right)\binom{2m-k}{k} = \frac{2m}{2m-k}\binom{2m-k}{k}.$$

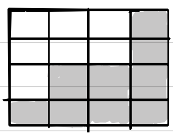
Theorem (Touchard) The following formula holds:

$$M_m = 2\cdot m! \cdot \sum_{k=0}^{m} (-1)^k (m-k)! \cdot \frac{2m}{2m-k} \binom{2m-k}{k}$$

Trivial by the previous considerations. □

Ferrers boards

We say an m-board is a **Ferrers board** if its subset consists of vertical piles of cells in weakly increasing order from left to right. Each Ferrers board is identified by a m-tuple $(a_1, \dots, a_m)$ with $0 \le a_1 \le \dots \le a_m$, where each a_i represents how many cells are in the i-th column.



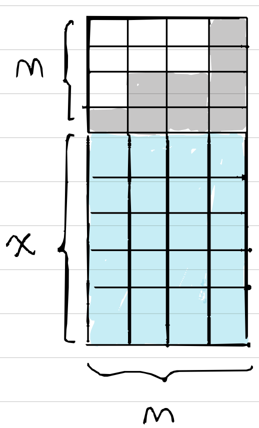
$(1,2,2,4)$

Theorem (Goldman-Joichi-White) Let a_i be s.t. $0 \le a_1 \le \dots \le a_m$. Then the following formula holds:

$$\prod_{i=1}^{m} (x + a_i - i + 1) = \sum_{k=0}^{m} r_{m-k}(F)\, (x)_k,$$

where F is the Ferrers board identified by $(a_1, \dots, a_m)$.

We prove the equality by double counting for $x > m$ (then the diff of the two polynomials will have infinite roots, hence they will be equal).

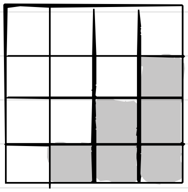


Let's add x rows below F. We can count the conf's of m-non att. rooks in $F + x$ in two ways:

- $\prod_{i=1}^{m} (x + a_i - i + 1)$: for each column i we have $x + a_i$ choices in the pile, but $i-1$ rows were already chosen.

- $\sum_{k=0}^{m} r_{m-k}(F)\,(x)_k$: we choose a conf. in F with $m-k$ rooks, then we just need k more rooks in x, for which we have $(x)_k$ choices. □

Let SS_n be the solid staircase board, i.e. the Ferrers board $(0,1,\dots,n-1)$.



SS_4

From the previous Theorem we get:

$$x^n = \sum_{k=0}^{n} r_{n-k}(SS_n)\,(x)_k .$$

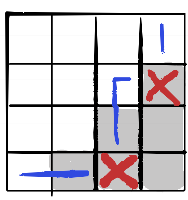
We also already knew that:

$$\underbrace{x^n}_{\substack{\text{func. from} \\ [n] \text{ to } [x]}} = \sum_{k=0}^{n} \underbrace{\left\{ {n \atop k} \right\}}_{\substack{\text{pre-images} \\ \text{with } k \\ \text{images}}} \underbrace{(x)_k}_{k \text{ images}}$$

Since $\{(x)_k\}$ forms a basis, we instantly get:

$$\boxed{r_k(SS_n) = \left\{ {n \atop n-k} \right\}} \quad (2)$$

$\rightsquigarrow$ a bijection between k-partitions of n and k-conf's of SS_n is given as follows:



$\{\{1,3,4\},\{2\}\}$

- given a k-conf in SS_n, we give the following relation:

  $i \sim j \iff \exists$ X s.t. j is above X and i on its left or viceversa

  The quotient by the transitive and reflexive closure of such relation gives the desired partition;

- there's an inverse process to the operation done above.

Eulerian polynomials

Let $\sigma \in S_n$. We say $i \in [n]$ is a rise (strict rise) of σ if $\sigma(i) \geq i$ ($\sigma(i) > i$); we say i is a fall if $\sigma(i) < i$ (i.e., if it's not a rise).

Let $\text{rise}(\sigma) \triangleq \# \{ i \mid i \text{ rise for } \sigma \}$, $\text{srise}(\sigma) \triangleq \{ \text{" | " strict "} \}$ and $\text{fall}(\sigma) \triangleq \# \{ \text{" | " fall "} \}$.

$\rightsquigarrow$ $\text{rise}(\sigma) + \text{fall}(\sigma) = n$.

We say $E_n(t) \triangleq \sum_{\sigma \in S_n} t^{\text{rise}(\sigma)}$ is the n-th Eulerian polynomial, whose coeff's count permutations with a fixed number of rises.

$\rightsquigarrow$ The reciprocal polynomial of $E_n(t)$, i.e., $t^n E(1/t)$ counts falls:

$$t^n E_n(1/t) = \sum_{\sigma \in S_n} t^{\text{fall}(\sigma)}$$

$E_n(t)$ is exactly the hit polynomial of $\overline{SS_n}$ (the compl. of SS_n):

$$E_n(t) = H(t, \overline{SS_n}),$$

hence $t^n E_n(1/t) = H(t, SS_n)^*$. By rotating SS_n by $180°$, we get the board whose hit polynom. is $\sum_{\sigma \in S_n} t^{\text{srise}(\sigma)}$. Hence, since this has the same hit polynomial of SS_n, we get:

$$\sum_{\sigma \in S_n} t^{\text{srise}(\sigma)} = \sum_{\sigma \in S_n} t^{\text{fall}(\sigma)} = t^n E_n(1/t)$$
$$= H(t, SS_n) \qquad (3)$$

* in general $\underbrace{t^n H(1/t, A)}_{\text{reciproc. pol. of } H(T,A)} = H(t, \overline{A})$.

$\rightsquigarrow$ There always exists a rise in $\sigma \in S_m$; otherwise we'd have $\sigma(i) < i \;\; \forall i \Longrightarrow$

$\Longrightarrow \;\; i > \sigma(i) > \sigma^2(i) > \dots > \sigma^m(i)$. By the pigeonhole principle, there's a $1 \le j \le m$ s.t. $\sigma^j(i) = i$, hence we'd have $i > i$, $\lightning$.

Therefore $t \mid E_m(t)$

Let σ^* be $\sigma + 1$ (i.e., $\sigma^*(i) =$ "$\sigma(i) + 1 \mod m$" $= \begin{cases} 1 & \text{if } \sigma(i)+1 = m+1 \\ \sigma(i)+1 & \text{otherwise} \end{cases}$). We observe that:

$$\operatorname{srise}(\sigma^*) = \operatorname{rise}(\sigma) - 1,$$

Therefore, by (1), (2) and (3) ($\cdot^*$ is invertible):

$$H(t, SS_m) = \sum_{\sigma^* \in S_m} t^{\operatorname{srise}(\sigma^*)} = \sum_{\sigma \in S_m} t^{\operatorname{rise}(\sigma) - 1} = E_m(t)/t \Longrightarrow$$

$$\Longrightarrow \boxed{E_m(t) = t\, H(t, SS_m) = t \sum_{k=0}^{m} (m-k)! \left\{ {m \atop m-k} \right\} (t-1)^k}$$