The symbolic method

We want to skip going through recursions and go straight to a functional equation or a gen. fun.

We were used to look at $\{A_m\}_{m\in\mathbb{N}}$, $a_m = |A_m|$ and $f(x) = \sum_{m\in\mathbb{N}} a_m x^m$.

Let's now look at $A = \bigcup_{m\in\mathbb{N}} A_m$ with $s: A \to \mathbb{N}$, which is called a STATISTIC. We're then interested in $f(x) = \sum_{a\in A} x^{s(a)}$ (e.g., Eulerian polynomial).

like a "weight" — a sign can also be introduced

We can generalise this method to multiple indices by considering multiple statistics $s_1, \dots, s_k$ and
$$f(x_1, \dots, x_k) = \sum_{a\in A} x_1^{s_1(a)} \cdots x_k^{s_k(a)}$$

→ $s: A \to \mathbb{N}$ must have finite pre-images to ensure $f(x)$ is actually a formal power series.

→ we can also use a comm. ring R instead of $\mathbb{N}$!

Example (Fibonacci numbers)

$A_m = \{\text{words in } \Sigma = \{1,2\} \text{ summing to } m\}$, $F_0 = 0$, $F_{m+1} = |A_m|$.

Let's rephrase the problem:

$$1 \longleftrightarrow \square, \qquad 2 \longleftrightarrow \boxed{}$$

→ e.g. $1211 \longleftrightarrow \square\boxed{}\square\square$ e.g., $s(1211) = 5$

Let $s: \underbrace{A = \bigcup_{m\in\mathbb{N}} A_m}_{\Sigma^*} \to \mathbb{N}$ s.t. $s(w) =$ "length of the word with weights" $=$

$\sum_{\sigma\in\Sigma} \sigma = \sum_{i\geq 0} (\square + \boxed{})^i$ – where $\cdot$ is the concaten.

Hence $f(x) = \sum_{\sigma\in\Sigma} x^{s(\sigma)}$. Each word is generated by the product $\square$, $\boxed{}$, thus:

$$f(x) = 1 + (x+x^2) + (x+x^2)^2 + \dots = \frac{1}{1-x-x^2}$$

→ The method that we just used for Fibonacci numbers can be adapted to any instance of $A = \Sigma^*$ given weights for Σ (which extend naturally to Σ^*):

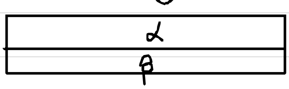
$$\Sigma = \{x_1, \dots, x_m\}$$

$$\Downarrow$$

$$\left[\ \sum_{\sigma \in \Sigma^*} x^{s(\sigma)} = \frac{1}{1-(x^{s(x_1)} + \dots + x^{s(x_m)})}\ \right]$$

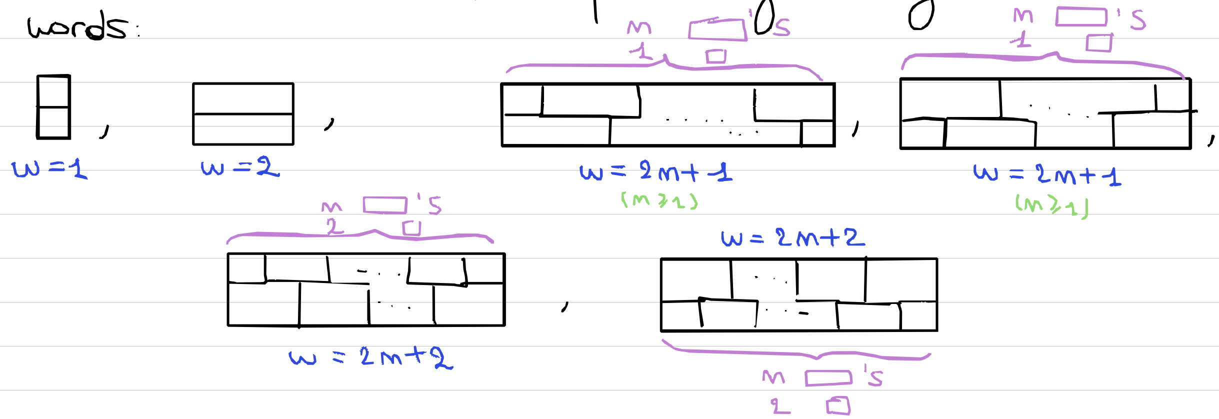
Example $B_m = \{(\alpha, \beta) \mid \alpha, \beta \in \underbrace{A_m}_{\text{def. as before}}\}$ $\qquad |B_m| = |A_m|^2 = F_{m+1}^2$

We can regard any $(\alpha, \beta) \in A_m^2$ as a word on "two layers":



Let's compute the building blocks of the alphabet generating these

words:



Thus:

$$\sum_{i \geq 0} F_{m+1}^2 x^m = \frac{1}{1-(x+x^2+2\sum_{m\geq 1} x^{2m+1} + 2\sum_{m \geq 0} x^{2m+2})} =$$

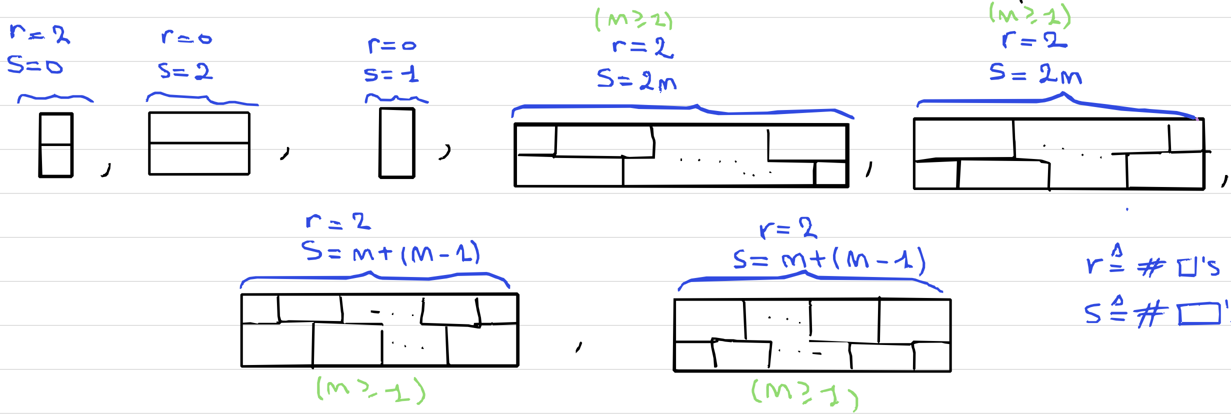
$$= \frac{1}{1-x-x^2-2\sum_{m\geq 2} x^m} = \frac{1}{1-x-x^2-\frac{2x^2}{1-x}} =$$

$$= \frac{1-x}{1-x-\cancel{x^2}-x+\cancel{x^2}+x^3-2x^2} = \frac{1-x}{1-2x-2x^2+x^3} =$$

$$= \frac{1-x}{(1+x)(1-3x+x^2)} \quad ✓$$

Example 2  We have $m \times 2$ strip we want to fill with $\square$, $\boxed{\quad}$ → also vertically

The generating set is the same as that of B_m's, except for the addition of $\square$:



$r \triangleq \#\ \square$'s

$s \triangleq \#\ \boxed{\quad}$'s

Thus:

$$\sum_{\substack{\sigma \in B \\ \text{all tilings}}} y^{r(\sigma)} z^{s(\sigma)} = \left[1-\left(y^2+z^2+z+2\sum_{m\geq 1} y^2 z^{2m}+2\sum_{m\geq 1} y^2 z^{2m-1}\right)\right]^{-1}$$

$$= \dots = \frac{1-z}{(1-z-z^2)(1-z)-(1+z)y^2}.$$

Example (partitions) $m \in \mathbb{N}$. Recall a partition of m is a vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \dots \geq \lambda_k > 0$ s.t. $\sum_i \lambda_i = m$. We say $\ell(\lambda) \triangleq k$ is its length, while m is its size.

Recall a partition is identified by one Ferrers diagram, which is the same as a Ferrers board, except for the fact that we don't draw boxes outside of our hit board and we rotate the board by 90° clockwise.

$\lambda = (4,3,2,2) \vdash 11$

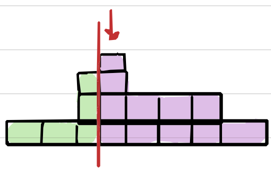
In this case the building set is $\left\{ \underbrace{\boxed{\;|\;\cdots\;|\;}}_{i \text{ times}} \right\}_{i \geq 1}$ and we're looking for the concatenation of such blocks:

$$\sum_{m \geq 1} B(m)\, x^m = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

→ The "infinite product" is interpreted as follows:

$$B(m) = [x^m] \sum_{m \geq 1} B(m)\, x^m = [x^m] \prod_{i \geq 1}^{m} \frac{1}{1 - x^i}.$$

Example (Stack polyominoes) Let's look at a generalization of dominoes: STACK POLYOMINOES.



If we consider the left-most cube, we can cut the polyominoes in two Ferrers diagrams:

There must be one col. on length m

$$f(x,y,q) = \sum_{\underbrace{P \in \text{Stack}P}_{\text{polyom.}}} x^{\text{rows}(P)}\, y^{\text{cols}(P)}\, q^{\text{area}(P)} = \sum_{m \geq 1} \underbrace{\frac{x^m y\, q^m}{\prod_{i=1}^{m} (1 - y q^i)}}_{\text{cols in pink}} \underbrace{\frac{1}{\prod_{i=1}^{m-1} (1 - y q^i)}}_{\text{cols in green}}$$

length of the col. with left-most point

Hence $\#\{\text{polyom. with } k \text{ blocks}\} = [q^k]\, f(1,1,q) =$

$$= [q^k] \sum_{m \geq 1} \frac{q^m}{\prod_{i=1}^{m} (1 - q^i) \prod_{i=1}^{m-1} (1 - q^i)}$$