Symmetric polynomials

p fixed point $\longleftrightarrow$ $\mathrm{Stab}(p) = S_m$

Let $k \in \mathbb{N}$. $S_k \curvearrowright \mathbb{Q}[x_1, \dots, x_k]$ s.t.

$$\sigma \circ f(x_1, \dots, x_m) = f(x_{\sigma(1)}, \dots, x_{\sigma(m)}).$$

Its fixed points are called SYMMETRIC POLYNOMIALS, whose set is denoted as $\mathrm{Sym}[X_k]$ $\quad X_k \triangleq \{x_1, \dots, x_k\}$

it's a sub-algebra of $\mathbb{Q}[X_k]$.

$\rightsquigarrow$ $\mathrm{Sym}[X_k] = \bigoplus_{m \geq 0} \mathrm{Sym}^m[X_k]$ } hom. symm. poly. in X_k

- We define $\left[e_m \triangleq e_m(x_1, \dots, x_k) \triangleq \sum_{1 \leq i_1 < \dots < i_m \leq k} x_{i_1} \cdots x_{i_m}, \right]$
  $e_0 \triangleq 1$.

$\rightsquigarrow$ for $m > k$, e_m is zero (empty sum).

e_i's are called the ELEMENTARY SYMM. POLY.

$\rightsquigarrow$ Viète's formulas hold: $a_k = (-1)^{m-k} e_{m-k}(\alpha_1, \dots, \alpha_m)$ } for $p(x) \triangleq \prod^m (x - \alpha_i) = \sum^m a_i x^i$

- We define $h_m \triangleq h_m(x_1, \dots, x_n) \triangleq \sum_{1 \leq i_1 \leq \dots \leq i_m \leq k} x_{i_1} \cdots x_{i_m}$ as the (COMP.) HOMOGENEOUS POLYNOMIAL in X_k OF DEGREE m.

$\rightsquigarrow$ h_m does <u>not</u> vanish, e_m does for $m > k$!

Recall the following theorem:

Theorem (fund. theorem of sym. poly.) $\underbrace{\mathrm{Sym}[X_k] \cong \mathbb{Q}[e_1, \dots, e_k]}_{\text{iso. of graded algebras}}$.

$\rightsquigarrow$ an equivalent statement is: $\{e_\lambda\}_{\lambda \vdash m}$ is a basis of $\mathrm{Sym}^m[X_k]$ where $e_\lambda \triangleq e_{\lambda_1} \cdots e_{\lambda_\ell}$ with $\underbrace{\lambda = (\lambda_1, \dots, \lambda_\ell)}_{\text{partition}}$

We're interested in having infinitely many variables. $X \triangleq \{x_1, \dots\}$ countable inf.

Def. $\mathrm{Sym}^m[X] \triangleq \{$ form. pow. series in $x_1, \dots$ with coeff. in $\mathbb{Q}$ that are symmetric and homog. of degree $m\}$

degree of monomials is bounded!

Def. $\underbrace{\mathrm{Sym}[X] \triangleq \bigoplus_{m \in \mathbb{N}} \mathrm{Sym}^m[X]}$. $\longrightarrow$ SYMMETRIC FUNCTIONS

$\rightsquigarrow$ $e_m \triangleq e_m(X) \triangleq \sum_{1 \le i_1 < i_2 < \dots < i_m} x_{i_1} \cdots x_{i_m} \longrightarrow$ ELEM. SYM. FUNCTIONS

$\rightsquigarrow$ $h_m \triangleq h_m(X) \triangleq \sum_{1 \le i_1 \le i_2 \le \dots \le i_m} x_{i_1} \cdots x_{i_m} \longrightarrow$ (COMP.) HOMOG. FUNCTIONS

Similarly we get:

iso. of graded algebras

Theorem (fund. Theo. of sym. func.) $\mathrm{Sym}[X] \cong \mathbb{Q}[e_1, \dots]$

Let $H(t) \triangleq \sum_{m \ge 0} h_m t^m = \prod_{i \ge 1} 1/_{1 - x_i t}$.

$\rightsquigarrow$ $E(t) \triangleq \sum_{m \ge 0} e_m t^m = \prod_{i \ge 1} (1 + x_i t)$.

Hence $H(t) \cdot E(-t) = 1 \rightsquigarrow \left[\sum_{i=0}^{m} (-1)^{m-i} e_{m-i} \cdot h_i = \delta_{0,m} \right]$

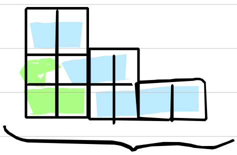
Moreover $\sum_{m \ge 0} h_m t^m = \dfrac{1}{\sum_{m \ge 0} (-1)^m e_m t^m}$ $\rightsquigarrow$ starts with 1 (thus it's a geo. series)

$\rightsquigarrow$ for $\lambda \vdash m$ we define h_λ in the same way we def. e_λ.

Q. We know $\{e_\lambda\}_{\lambda \vdash m}$ is a basis for $\mathrm{Sym}^m[X]$, thus how do we write h_ν $(\nu \vdash m)$ in terms of those elements?

$$h_\nu = \sum_{\lambda \vdash m} ? \; e_\lambda.$$

In order to do so we will use the set of BRICK TABLEAUX of type λ.



a brick Tableaux of Type $\lambda = (4,3,2,2,1)$ and shape $\mu = (6,4,2)$

"These are refinements of partitions"

$B_{\lambda,\mu} = \{$brick Tableaux of Type λ and shape $\mu\}$

shape μ = Ferrers diagram of μ

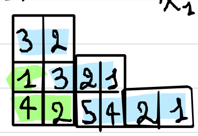
Type λ: each el. of λ is identified by a horiz. block in the diagram

Theorem $\quad h_\mu = \sum_{\lambda \vdash m} (-1)^{m-\ell(\lambda)} |B_{\lambda,\mu}| \, e_\lambda \quad$ (1) for $\mu \vdash m$

$\lambda = (\lambda_1, \dots, \lambda_k) \Longrightarrow \ell(\lambda) = k$

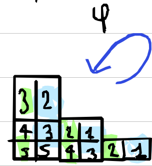
Consider The set of bricks tab. of shape μ in which, for each sub partition, we add numbers in a strongly decreasing order.

$(-1)^{m-\ell(\lambda)+1} x_1^3 x_2^4 x_3^2 x_4^2 x_5$

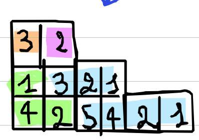


λ, μ

φ



(fixed)



$(-1)^{m-\ell(\lambda)+1} x_1^3 x_2^4 x_3^2 x_4^2 x_5$

Now we assign to each such object the weight:

$$(-1)^{m-\ell(\lambda)} \prod x_i^{\#\text{Times } i \text{ appears}}$$

Let φ be the following sign-rev. involution on such a set:

- if you first cross a brick of len $k \geq 2$, you cut it s.t. you have now a 1-block and a $(k-1)$-block
- if you first cross a 1-block $\boxed{t}$ foll by a k-block $\boxed{t' \; \dots}$ s.t. $t > t'$, you merge them.

Thus we can count only the elements fixed by φ (the others vanish); i.e. the boards with only 1-blocks in weakly decr. order in the row.

Thus:

$$\sum_{B\in \text{fix}\,\varphi} w(B) = \sum_{B\in \text{boards}} w(B)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\prod_{N_J\in N}\Big(\sum_{1\le i_1\le\cdots\le i_n\le N_J} x_{i_1}^{\lambda_1} x^{\lambda_2}\cdots\Big) \qquad\qquad \sum_{\lambda\vdash m}\ \sum_{\substack{\text{positionings}\\ \text{of }\lambda\text{ with}\\ \text{labels in }N}} (-1)^{m-\ell(\lambda)} \prod x_\lambda^{\#\cdots}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\prod_{N_J\in N} h_J = h_N \qquad\qquad \sum_{\lambda\vdash m}(-1)^{m-\ell(\lambda)}\underbrace{|B_{\lambda,N}|}_{\substack{\prod\text{ choices for}\\ \lambda_i\in\lambda\text{ block }\lambda_i}} e_\lambda$$

∎

Eulerian polynomials (again)

We re-set $E_m(t) \triangleq \sum_{\sigma\in S_m} t^{\text{fall}(\sigma)}$, since we observed that the "old $E_m(t)$" was divisible by t.

We say $i\in[m-1]$ is a **DESCENT** for $\sigma \in S_m$ iff $\sigma(i)>\sigma(i+1)$ and set

$$\text{des}(\sigma) = \#\text{ descents for }\sigma.$$

Prop. $E_m(t) = \sum_{\sigma\in S^m} t^{\text{des}(\sigma)}$

increasing order!

Let $\varphi: S^m \to S^m$ s.t. it takes the cycle not. of σ, it orders cycles by the greatest number and sets $\varphi(\sigma)$ the perm. $\sigma^* =$ one-line not. of what we just obtained.

φ is not an involution!! It sends falls into descents, but not viceversa!

$$\sigma = 54231 = (1,5)(2,4,3) = (4,3,2)(5,1)$$
$$\in S_5$$

$$\sigma^* = 43251 \xleftarrow{\ \varphi\ } \qquad (4,3)(5,2,1)$$

This map is a bijection and is s.t. $\text{falls}(\sigma) = \text{des}(\underbrace{\varphi(\sigma)}_{=\sigma^*})$. ∎

Let $\varphi : \mathrm{Sym}[X] \longrightarrow \mathbb{Q}[t]$ s.t. $\varphi(e_m) = (-1)^{m-1}(t-1)^{m-1}/m!$

(recall $\mathrm{Sym}[X] \cong \mathbb{Q}[e_1, \dots]$).

Theorem. $\varphi(h_m) = E_m(t)/m!$

m times

$$m!\,\varphi(h_m) \overset{(1)}{=\!=} m! \sum_{\lambda \vdash m} (-1)^{m-\ell(\lambda)} |B_{\lambda,(m)}|\, \varphi(e_\lambda) =$$

$$= m! \sum_{\lambda \vdash m} (-1)^{m-\ell(\lambda)} |B_{\lambda,(m)}| \prod_{\lambda_i \in \lambda} (-1)^{\lambda_i - 1} (t-1)^{\lambda_i - 1} / \lambda_i! =$$

$$= \sum_{\lambda \vdash m} \underbrace{\binom{m}{\lambda_1 \dots \lambda_m}}_{\text{in a comb. way}} \underbrace{|B_{\lambda,(m)}|}_{\text{for each iter.}} (t-1)^{m-\ell(\lambda)} = (*)$$

Let's interpret the sum. It fixes a layout of type λ ($|B_{\lambda,(m)}|$) for (m), chooses the numbers to put into each box and, on a second row, chooses for each square between t and -1 — except for right-most squares (for which 1 is mandatory).

$\binom{m}{\lambda_1 \dots \lambda_m}$

$1 \updownarrow (\varphi)$

1	t	t	1	-1	1	-1	t	1
6	7	3	1	5	2	9	8	4

To each such object we assign then the weight:

$$(-1)^{\#-1\text{'s}}\, t^{\#t\text{'s}}$$

Let φ the sign-rev. involution s.t.

(i) if you first cross a -1, cut the block there and swap -1 with 1

(ii) if you first cross a 1 block s.t. the label is a descent, merge the blocks and swap 1 with -1.

Hence $(*) = \sum_{\sigma \in \mathrm{fix}\,\varphi} w(\sigma)$ and fixed points are exactly those which count descents,

Thus $(*) = E_m(t)$ ✓ $\square$

Corollary $\sum_{m\in\mathbb{N}} E_m(t)\frac{z^m}{m!} = (t-1)/(t-e^{z(t-1)})$.

$$\sum_{m\in\mathbb{N}} E_m(t)\frac{z^m}{m!} = \sum_{m\in\mathbb{N}} \varphi(h_m)\, z^m \overset{\bullet}{=}$$

$H(z)E(-z) = 1$

$m!\,\varphi(h_m) = E_m(t)$

$$\overset{\bullet}{=} \left(\sum_{m\in\mathbb{N}} (-1)^m \varphi(e_m)\, z^m\right)^{-1} =$$

$$= \left(1 + \sum_{m\geq 1} (-1)^m (-1)^{m-1} (t-1)^{m-1} \frac{z^m}{m!}\right)^{-1} =$$

$$= (t-1)/(t-e^{z(t-1)}).$$

✓ □