Formula sheet for Algebraic combinatorics

Formula 1 (Sum of two power series).

$$\sum_{i\geq 0} a_i x^i + \sum_{i\geq 0} b_i x^i \triangleq \sum_{i\geq 0} (a_i + b_i) x^i.$$

Formula 2 (Sum of multiple power series).

$$\mathop{+}_{i\in[n]} \sum_{j\geq 0} a_{i,j}\, x^j = \sum_{j\geq 0} \left(\sum_{i\in[n]} a_{i,j} \right) x^j.$$

Formula 3 (Product of two power series).

$$\left(\sum_{i\geq 0} a_i x^i\right) \cdot \left(\sum_{i\geq 0} b_i x^i\right) \triangleq \sum_{i\geq 0} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

Formula 4 (Product of multiple power series).

$$\prod_{i\in[n]} \sum_{j\geq 0} a_{i,j}\, x^j = \sum_{j\geq 0} \left(\sum_{k_1+\ldots+k_n=j} a_{1,k_1} \cdots a_{n,k_n} \right) x^j.$$

Formula 5 (Geometric series).

$$\frac{x^k}{(1-x)} = \sum_{i\geq k} x^i \in \mathbb{C}[[x]].$$

Prove the formula for $k = 1$ and then group x's to retrieve the general formula.

Formula 6 (Exponential series).

$$e^x \triangleq \sum_{i\geq 0} \frac{x^i}{i!}.$$

Formula 7 (Logarithmic series).

$$\log(1+x) \triangleq \sum_{i\geq 1} (-1)^{i+1} \frac{x^i}{i}.$$

It's "$\int 1/(1+x)\,\mathrm{d}x$".

Formula 8.

$$\log\left(\frac{1}{1-x}\right) = -\log(1-x) = \sum_{i\geq 1} \frac{x^i}{i}.$$

Formula 9 (Binomial coefficient).

$$\binom{n}{k} \triangleq \#\{B \subseteq [n] \mid |B| = k\}.$$

Number of ways for choosing k elements out of n.

Formula 10.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad n \geq 1.$$

You either choose n or you don't.

Formula 11 (Newton's binomial theorem).

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Apply Formula 10 using induction.

Formula 12.

$$\#\{B \subseteq [n]\} = 2^n.$$

Apply Formula 11 with $x = 1$. Alternatively, each subset B is uniquely identified by its characteristics function 1_B, hence the subsets of $[n]$ are counted by the functions from $[n]$ to $[2]$.

Formula 13.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Apply Formula 11.

Formula 14 (Formula for the binomial coefficient).

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Apply Formula 11 and derive $(1+x)^n$ k times.

Formula 15 (Falling factorials).

$$(x)_k \triangleq x(x-1)\cdots(x-k+1) = \prod_{i=0}^{k-1} (x-i), \quad x \in \mathbb{C}.$$

Formula 16 (Rising factorials).

$$x^{(k)} \triangleq x(x+1)\cdots(x+k-1) = \prod_{i=0}^{k-1} (x+i), \quad x \in \mathbb{C}.$$

Formula 17 (Binomial coefficients with $n \in \mathbb{C}$).

$$\binom{n}{k} \triangleq \frac{(n)_k}{k!}, \quad n \in \mathbb{C}, k \in \mathbb{N}.$$

This is compatible with how binomials were previously defined.

Formula 18 (Newton's binomial theorem for falling factorials).

$$(a+b)_n = \sum_{i=0}^{n} (a)_i (b)_{n-i}.$$

By induction on n.

Formula 19 (Order of a formal power series).

$$\operatorname{ord}(f(x)) \triangleq \operatorname{mdeg}(f(x)) \triangleq \min\{i \mid a_i \neq 0\}.$$

Formula 20 (Existence of k-roots in $\mathbb{C}$-power series).
$f(x)$ admits a k-root in $\mathbb{C}[[x]]$ if and only if $k \mid \operatorname{ord}(f(x))$.

Formula 21 (k-roots of $(1+x)$).

$$(1+x)^{1/k} = \sum_{i \geq 0} \binom{1/k}{i} x^i.$$

Apply Formula 18).