The inclusion-exclusion principle

Let $A_1, \dots, A_m$ be subsets of X. Let $a_S \triangleq \left|\bigcap_{J \in S} A_J\right|$ for $S \subseteq [m]$, $S \neq \emptyset$. Let $a_\emptyset = |X|$.

Theorem (The inclusion-exclusion principle) The two following formulas are valid:

(i) $$\left|X \setminus \bigcup_{i=1}^{m} A_i\right| = \sum_{S \subseteq [m]} (-1)^{|S|} a_S.$$

(ii) $$\left|\bigcup_{i=1}^{m} A_i\right| = \sum_{\substack{S \subseteq [m] \\ S \neq \emptyset}} (-1)^{|S|-1} a_S.$$

$\rightsquigarrow$ one can prove (ii) by induction and retrieve (i) easily.

Note that $a_S = \sum_{x \in X} \underbrace{\prod_{J \in S} 1_{A_J}(x)}$ $\longrightarrow$ $1_{\bigcap_{J \in S} A_J}(x)$.

Since $\left|X \setminus \bigcup_{i=1}^{m} A_i\right| = \sum_{x \in X} \prod_{i=1}^{m} \left(1 - 1_{A_i}(x)\right)$, by expanding one gets the result immediately. $\square$

Example (Derangements)

Let's use the incl.-excl. principle to retrieve a formula for D_m directly.

Let $A_i \subseteq S_m$ the subset of permutations fixing $i \in [m]$. Then:

$$D_m = \left|S_m \setminus \bigcup_{i=1}^{m} A_i\right| = \sum_{S \subseteq [m]} (-1)^{|S|} a_S =$$

$$= \sum_{k=0}^{m} \sum_{\substack{S \subseteq [m] \\ |S| = k}} (-1)^k a_S = \sum_{k=0}^{m} (-1)^k \underbrace{\sum_{\substack{S \subseteq [m] \\ |S| = k}} \overbrace{a_S}^{= |S_{m-k}|}}_{= \binom{m}{k}} =$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)! = m! \left[\sum_{k=0}^{m} \frac{(-1)^k}{k!}\right]$$

Example (surjective functions from $[m]$ to $[k]$)

We already noticed that $\#\{\text{surj. fun. from } [m] \text{ to } [k]\} = k! \left\{ {m \atop k} \right\}$.

Let's use the incl.-excl. principle to retrieve this formula

Let $A_i \subseteq [k]^{[m]}$ be the set of functions not having $i \in [k]$ as an image. Hence a_S counts how many functions are s.t. $S \cap f([m]) = \emptyset$.

$$\rightsquigarrow \quad a_S = \left| [k] \setminus S^{[m]} \right| = (k - |S|)^m$$

Hence:

$$\#\{\text{surj. fun. from } [m] \text{ to } [k]\} = \left| [k]^{[m]} \setminus \bigcup_{i=1}^{k} A_i \right| =$$

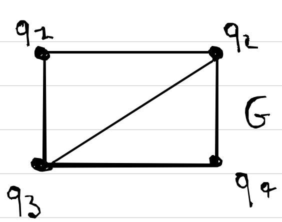
$$= \sum_{i=0}^{k} (-1)^k \sum_{\substack{S \subseteq [k] \\ |S| = k}} a_S =$$

$$= \sum_{i=0}^{k} (-1)^k \binom{k}{i} (k-i)^m \quad \checkmark$$

Example (proper colorings for a graph)

A **proper coloring** is a coloring s.t. adjacent nodes have different colors.

Let $\chi(G, q)$ be the number of proper q-colorings in G.

$\rightsquigarrow$ 

$$\chi(G, q) = \underbrace{q(q-1)(q-2)}_{\text{three diff. colors for } q_1, q_2, q_3} \underbrace{(q-2)}_{q_1 \text{ and } q_4 \text{ can have the same color}}$$

Let's use incl.-excl. to retrieve a formula for $\chi(G,q)$.

For each $e=\{u,v\}\in E$ we set:

$$A_e \triangleq \{\kappa \text{ q-coloring} \mid \kappa(u)=\kappa(v)\}$$

And:

$$X \triangleq \{\kappa \text{ q-colorings}\} \quad (|X| = q^{|V|})$$

Then:

$$a_S = \#\{\kappa \text{ q-colorings} \mid \kappa(s_1)=\kappa(s_2)\ \forall s_1,s_2 \in \cup S\} =$$

$$= \#\{\text{"} \mid \kappa \text{ constant on the connect. compon. of } (V,S)\} =$$

$$= q^{c(S)}, \quad \text{where } c(S) = \#\{\text{conn. comp. of } (V,S)\}$$

Hence:

$$\underbrace{\chi(G,q) = \sum_{S\subseteq E} (-1)^{|S|} q^{c(S)}}_{\text{chromatic polynomial of } G}$$

Generalisation of the inclusion-exclusion principle

Let $E_T \triangleq \{x\in X \mid x\in A_i \iff i\in T\}$ and $e_T \triangleq |E_T|$.

$\rightsquigarrow$ if A_i represents the "i-th property", E_T is the subset of X containing elements satisfying exactly the properties in T.

Let $a_r \triangleq \sum\limits_{\substack{S\subseteq X\\ |S|=r}} a_S$ and $e_r \triangleq \sum\limits_{\substack{S\subseteq X\\ |S|=r}} e_S$.

Theorem (general incl.-excl. formula) The following formula holds:

$$\sum_{k=0}^{m} e_k t^k = \sum_{k=0}^{m} a_k (t-1)^k \,,$$ where t is a formal variable.

$\rightsquigarrow$ for $t=0$, we get $|X \setminus \bigcup_i A_i| = e_0 = \sum_{k=0}^{m} (-1) a_k$, i.e., the usual incl.-excl. principle.

Let T be a subset of X. Then:

$$e_T = \sum_{x \in X} \left[\prod_{i \in T} \chi_{A_i}(x) \cdot \prod_{i \notin T} \chi_{\overline{A_i}}(x) \right]$$

Hence:

$$\sum_{k=0}^{m} a_k (t-1)^k = \sum_{k=0}^{m} \Big(\sum_{\substack{S \subseteq X \\ |S|=k}} a_S \Big) (t-1)^k =$$

$$= \sum_{S \subseteq X} a_S \, (t-1)^{|S|} =$$

$$= \sum_{S \subseteq X} \sum_{x \in X} \Big(\prod_{i \in S} \chi_{A_i}(x) \Big) (t-1)^{|S|} =$$

$$= \sum_{x \in X} \sum_{S \subseteq X} \Big(\prod_{i \in S} \chi_{A_i}(x) \Big) (t-1)^{|S|} =$$

think of $(t-1)\chi_{A_i}(x)$ as x_i and apply $\prod_i (1+x_i) = \sum_{S \subseteq I} \prod_{i \in S} x_i$

$$= \sum_{x \in X} \prod_{i=1}^{m} \big(1 + (t-1)\chi_{A_i}(x)\big) =$$

use $\chi_A + \chi_{\overline{A}} = 1$

$$= \sum_{x \in X} \prod_{i=1}^{m} \big(t\chi_{A_i}(x) + \chi_{\overline{A_i}}(x) \big) =$$

apply $\prod_{i \in I} (x_i + y_i) = \sum_{S \subseteq I} \Big[\prod_{i \in S} x_i \prod_{i \notin S} y_i \Big]$

$$= \sum_{x \in X} \sum_{T \subseteq X} \prod_{i \in T} \chi_{A_i}(x) \prod_{i \notin T} \chi_{\overline{A_i}}(x)\, t^{|T|} =$$

$$= \sum_{k=0}^{m} e_k t^k \qquad \square$$